Stability of an equilibrium point:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

$y_0$ is a **sink** if any solution with initial condition close to $y_0$ tends toward $y_0$ as $t$ increase.

$y_0$ is a **source** if any solution with initial condition close to $y_0$ tends toward $y_0$ as $t$ decrease.

$y_0$ is a **node** if it is neither a sink nor a source.
Linearization Theorem:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

- if $f'(y_0) < 0$, then $y_0$ is a sink;

- if $f'(y_0) > 0$, then $y_0$ is a source;

- if $f'(y_0) = 0$, then $y_0$ can be any type, but in addition
  - if $f''(y_0) > 0$ or $f''(y_0) < 0$, then $y_0$ is a node.
**Bifurcation**: Suppose that the differential equation depends on a parameter. Then we say that a bifurcation occurs if there is a qualitative change in the behavior of solutions as the parameter changes.

**Example 1**: \( \frac{dy}{dt} = ky(1-y) \) (no bifurcation)

**Example 2**: \( \frac{dy}{dt} = y^2 - \mu \) (saddle-node bifurcation, supercritical)

**Example 3**: \( \frac{dy}{dt} = y^3 + \mu y \) (pitchfork bifurcation, subcritical)

**Example 4**: \( \frac{dy}{dt} = y^2 - \mu y \) (transcritical bifurcation)
Example 5: Constant yield harvesting

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) - h
\]

Mathematical Analysis:
1. **Nondimensionalization**: \( u = \frac{P}{N}, s = kt, \)

\[
\frac{du}{dt} = u(1 - u) - H, \quad H = \frac{h}{kN}
\]

2. **Bifurcation**: a subcritical saddle-node bifurcation occurs at \( H = 0.25, \) or \( h = 0.25kN. \)

3. **Qualitative analysis**: when \( 0 < H < 0.25, \) \( H = 0.25 \) and \( H > 0.25. \)

4. **Analytic method**: solve the equation? (see homework)
Example 5 (Cont.): Constant yield harvesting

\[ \frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) - h \]

Biological interpretation:
1. When \( 0 < H < 0.25 \), there are two equilibrium points \( P_1 > P_2 > 0 \); for \( P(0) > P_2 \), \( \lim_{t \to \infty} P(t) = P_1 \) and for \( 0 < P(0) < P_2 \), \( P(t) < 0 \) for \( t > t_0 \); \( P_1 \) is smaller than \( N \), that means the carrying capacity decreases because of harvesting; the behavior of solutions with \( P(0) > P_2 \) is similar to that of logistic equation; if the initial population is less than \( P_2 \), then the population becomes extinct in finite time.
2. When \( H > 0.25 \), there is no equilibrium points, and for any initial population, the population becomes extinct in finite time.
3. \( H = 0.25 \) or \( h = 0.25kN \) is called Maximum Sustainable Yield (MSY).
Example 6: Constant effort harvesting

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) - hP
\]

Mathematical Analysis:

1. **Nondimensionalization**: \( u = \frac{P}{N}, s = kt, \)

   \[
   \frac{du}{dt} = u(1 - u) - Hu, \quad H = \frac{h}{kN}
   \]

2. **Bifurcation**: no bifurcation if we assume that \( 0 < H < 1 \)

3. A different question: for which \( H \), we can get the maximum yield \( Hu \)?