Summary of Part 1:

Population models for single species:

Model 1: Malthus model
Model 2: Logistic model
Model 3a,b: Allee effect
Model 4a,b,c,d,e: Harvesting model
Model 5: Spruce budworm model

Fishing model: management of renewable natural resources
(fish, forest...)

1
A General Population Model: (deterministic model)

\[
\frac{dP}{dt} = [B(t) - D(t)]P \left[1 - \frac{P}{N(t)}\right] + I(t) - E(t)
\]

*B(t):* birth rate per capita at \( t \)

*D(t):* death rate per capita at \( t \)

*N(t):* carry capacity due to technology

*I(t):* rate of immigration

*E(t):* rate of emigration

Stochastic model: \( P(t) \) not a function, but a probability distribution
Mathematical methods for single ODE:

1. Analytic methods: separation of variables.


3. Qualitative methods: direction field, graph of solutions, phase line, equilibrium points and their stability, bifurcation.

**Model 4d:** Holling’s type II model

\[
\frac{dQ}{ds} = Q (1 - Q) - \frac{hQ}{1 + aQ}
\]

**A.** when \( a = 2 \), \( Q (1 - Q) - \frac{hQ}{1 + 2Q} = 0 \)

\( Q = 0 \) or \( 2Q^2 - Q + (h - 1) = 0 \), \( Q = \frac{1 \pm \sqrt{9 - 8h}}{4} \)

Bifurcation diagram: \( h = -2Q^2 + Q + 1 \) (but in a \( h - Q \) graph)

Two bifurcation points:
\( h = 9/8 \): subcritical saddle-node bifurcation
\( h = 1 \): transcritical bifurcation
Biological implications when $a = 2$:

1. When $0 < h < 1$, there are two equilibrium points, 0 and $Q_+ = 0.25(1 + \sqrt{9 - 8h})$. The system is similar to logistic equation.

2. When $1 < h < 9/8$, there are three equilibrium points, 0 and $Q_\pm = 0.25(1 + \sqrt{9 - 8h})$. The system is similar to logistic equation with allee effect.

3. When $h > 9/8$, 0 is the only equilibrium point. The population will become extinct no matter how large the initial population is.

4. Depending on the value of $h$, the state of the system is survival, partial survival and extinction.
B. General \((a, h)\). \(Q (1 - Q) - \frac{hQ}{1 + aQ} = 0 \)

\[Q = 0 \text{ or } aQ^2 + (1 - a)Q + (h - 1) = 0,\]

\[Q_{\pm} = \frac{a - 1 \pm \sqrt{(a + 1)^2 - 4ah}}{2a}, \text{ Basic border line: } h = \frac{(a + 1)^2}{4a}\]

when \(0 < h < \frac{(a + 1)^2}{4a}\), three equilibrium points

when \(h = \frac{(a + 1)^2}{4a}\), two equilibrium points (except \(a = 1\))

when \(h > \frac{(a + 1)^2}{4a}\), one equilibrium points

But we also count the negative equilibrium points
Delicate but simple analysis:

when \( Q_+ < 0 \): \( Q_+ = \frac{a - 1 + \sqrt{(a + 1)^2 - 4ah}}{2a} < 0 \)

when \( Q_+ \geq 0 \) but \( Q_- < 0 \): \( Q_- = \frac{a - 1 - \sqrt{(a + 1)^2 - 4ah}}{2a} < 0 \)

Derivative method:

Solve \( f(Q) = aQ^2 + (1 - a)Q + (h - 1) = 0 \)
and \( f'(Q) = 2aQ + (1 - a) = 0 \).