On dispersal and population growth for multistate matrix models

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Here we want to study the change of the overall growth rate if dispersions take place in the m age classes governed by the $n \times n$ column stochastic matrices $S_1, \ldots, S_m$.

[The (p,q) entry of S_j represents the fraction of the individuals in the jth age class moving from patch q to patch p.]

The system is now described by $\mathbf{SD}$, where

$$\mathbf{S} = S_1 \oplus \cdots \oplus S_m \quad \text{and} \quad \mathbf{D} = (D_{ij})_{1 \le i,j \le m}$$

such that each $D_{ij} \in M_n$ is a diagonal matrix and

$$\mathbf{D}[1, n+1, \ldots, (m-1)n+1; 1, n+1, \ldots, (m-1)n+1] = A_1,$$

$$\mathbf{D}[2, n+2, \ldots, (m-1)n+2; 2, n+2, \ldots, (m-1)n+2] = A_2, \ldots,$$

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Example Suppose $S_1, S_2 \in M_3$, and $A_1, A_2, A_3 \in M_2$. Then

$$\mathbf{D} = \left(\begin{array}{ccc|ccc} (A_1)_{11} & 0 & 0 & (A_1)_{12} & 0 & 0 \\ 0 & (A_2)_{11} & 0 & 0 & (A_2)_{12} & 0 \\ 0 & 0 & (A_3)_{11} & 0 & 0 & (A_3)_{12} \\ \hline (A_1)_{21} & 0 & 0 & (A_1)_{22} & 0 & 0 \\ 0 & (A_2)_{21} & 0 & 0 & (A_2)_{22} & 0 \\ 0 & 0 & (A_3)_{21} & 0 & 0 & (A_3)_{22} \end{array}\right).$$

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How about the general case?

An Example

Consider a population of juveniles and mature adults living in two spatial locations (e.g., small salmon develop in fresh water and mature salmon move to ocean to reproduce). Suppose

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 100 \\ 0 & 0 \end{pmatrix}.$$

That is, each small salmon grows to a mature salmon in fresh water, and each salmon reproduces 100 salmon in the ocean. If no dispersion is allowed, i.e., $\mathbf{S} = I_2 \oplus I_2$, then the growth rate of the whole system equals

$$\rho(\mathbf{D}) = \rho \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) = 0.$$

The population goes extinct in two time steps.

Alternatively, if all juveniles move to patch 1 and all adults move to patch 2, then

$$S_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

In which case,

$$\rho(\mathbf{SD}) = \rho \begin{pmatrix} 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 10,$$

and the population grows asymptotically at a geometric rate.

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Question Determine $\mathbf{S}$ so that $\rho(\mathbf{SD}) \leq \rho(\mathbf{D})$ for all $\mathbf{D}$.

Result

Definition Suppose $\mathbf{S} = S_1 \oplus \cdots \oplus S_m$ is given. For $i = 1, \ldots, m$, let G_i be the directed graph of S_i with vertices $1, \ldots, n$. Here we ignore the diagonal entries of S_i, and assume that the edges of G_i has color i.

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We say that $S_1, \ldots, S_m$ admits a polychromatic cycle if there are nonzero entries of $S_1 + \cdots + S_m$ at the $(i_1, i_2), (i_2, i_3) \ldots, (i_{k-1}, i_k), (i_k, i_1)$ positions for some distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$, and these nonzero entries do not come from a single matrix S_j.

Theorem [Li and Schreiber] Suppose $\mathbf{S} = S_1 \oplus \cdots \oplus S_m$ so that $S_1, \ldots, S_m$ are $n \times n$ column substochastic matrices. The following conditions are equivalent.

(a) $S_1, \ldots, S_m$ admits a polychromatic cycle.

(b) There exists a block matrix $\mathbf{D} = (D_{ij})_{1 \leq i,j \leq m}$ such that $D_{ij} \in M_n$ is a diagonal matrix with positive (nonnegative) diagonal entries and such that $\rho(\mathbf{SD}) > \rho(\mathbf{D})$.

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Consequently, if one wants to ensure that $\rho(\mathbf{SD}) \leq \rho(\mathbf{D})$, and still allow dispersion, just make sure that $S_1, \ldots, S_m$ do not admit a polychromatic cycle!

Idea of Proof

Note that $\rho(\mathbf{SD}) = \rho(\mathbf{RA})$, where

$$\mathbf{A} = A_1 \oplus \cdots \oplus A_n \quad \text{and} \quad \mathbf{R} = (R_{ij})_{1 \le i,j \le n}$$

such that R_{ij} is a diagonal matrix and

$$\mathbf{R}[1, m+1, \ldots, (n-1)m+1; 1, m+1, \ldots, (n-1)m+1] = S_1, \ \ldots,$$

$$\mathbf{R}[2, m+2, \ldots, (n-1)m+2; 2, m+2, \ldots, (n-1)m+2] = S_2,$$

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(a) $\Rightarrow$ (b): Assume there $S_1, \ldots, S_m$ admits a polychromatic cycle. We cook up $\mathbf{D}$ strategically so that $\mathbf{D}$ is similar to $\mathbf{A} = \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_n$, where for each $j = 1, \ldots, n$, $\rho(\mathbf{A}_j) = 1$ and (some) $\mathbf{A}_j$ has large off-diagonal entries positioned in such a way that $\rho(\mathbf{SD}) > 1$.

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(b) $\Rightarrow$ (a): We assume that (a) is not true, i.e., $S_1, \ldots, S_m$ do not admit a polychromatic cycle. We show that there exists a nonnegative diagonal matrix $\mathbf{V}$ such that $\mathbf{VSV}^{-1}$ is column substochastic, and $\mathbf{VDV}^{-1}/\rho(\mathbf{D})$ is column stochastic. Thus, using the column sum norm $\|\cdot\|$, we have

$$\rho(\mathbf{SD}) = \rho(\mathbf{VSDV}^{-1}) \leq \|\mathbf{VSDV}^{-1}\| \leq \|\mathbf{VSV}^{-1}\| \|\mathbf{VDV}^{-1}\| \leq 1.$$

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The construction of $\mathbf{V}$ was done by an intricate labelling scheme of the vertices of $S_1, \ldots, S_m$ using the information of the Perron vectors of $A_1, \ldots, A_m$.

Further research

- Study $\mathbf{S}$ and/or $\mathbf{D}$ with special structures.

- Determine conditions on $\mathbf{S}$ and $\mathbf{D}$ so that $\rho(\mathbf{SD}) \leq \rho(\mathbf{D})$.

- For a pair of $(\mathbf{S}, \mathbf{D})$, study the monotonicity behavior of $\rho(\mathbf{S}(t)\mathbf{D})$, where $\mathbf{S}(t) = (1-t)I + t\mathbf{S}$.

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Thank you for your attention!!!