Numerical ranges, dilations, and norm estimations

Chi-Kwong Li
Department of Mathematics
College of William and Mary
Williamsburg, Virginia 23187-8795

Mainly from joint work with:
Man-Duen Choi (University of Toronto)

Notation and definitions

Let $\mathcal{H}$ be a Hilbert space.

Let $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$.

If $\mathcal{H}$ has dimension n, identify $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ with $(\mathbb{C}^n, M_n)$.

The numerical range of $A \in \mathcal{B}(\mathcal{H})$ is:

$$W(A) = \{(Ax, x) : x \in \mathcal{H},\ (x, x) = 1\}.$$

Examples

If $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $W(A) = [0, 1]$.

If $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ then $W(A)$ is the unit disk.

Basic properties

(a) $W(U^*AU) = W(A)$ for any unitary $U \in \mathcal{B}(\mathcal{H})$.

(b) $W(rA + sI) = rW(A) + s$.

(c) $W(A + E) \subseteq W(A) + W(E)$.

(d) $\sigma(A) \subseteq \overline{W(A)}$.

By (c) and (d), we have

$$\sigma(A + E) \subseteq \overline{W(A + E)} \subseteq \overline{W(A)} + \overline{W(E)}.$$

In general, $\sigma(A + E) \not\subseteq \sigma(A) + \sigma(E)$.

Example Let $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$. Then

$\sigma(A) = \sigma(E) = \{0\}$ and $\sigma(A + E) = \left\{\pm\sqrt{M\varepsilon}\right\} \subseteq W(A + E)$.

Some geometrical/topological properties

(a) $W(A)$ is bounded.

(b) $W(A)$ is closed if $\dim \mathcal{H} < \infty$.

(c) [Toeplitz-Hausdorff, 1918-19] $W(A)$ is always convex.

(d) If $A = U^* \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix} U$ then $W(A)$ is an elliptical disk with foci λ_1, λ_2, and minor axis with length $|b|$.

Remark To prove (d), Li [1996] showed that one needs only to verify the (easy) cases when

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

because $W(f(A)) = f(W(A))$ for any real affine transform f on $\mathbb{C}$ (identified as $\mathbb{R}^2$).

More useful properties

(a) $A = A^*$ if and only if $W(A) \subseteq \mathbb{R}$.

(b) $A = A^*$ is positive semi-definite if and only if

$$W(A) \subseteq [0, \infty).$$

(c) $A = \lambda I$ if and only if $W(A) = \{\lambda\}$.

(d) If A is normal, then $\overline{W(A)} = \operatorname{conv} \sigma(A)$. Furthermore, if $\sigma(A)$ is finite, then $W(A)$ is a convex polygon.

(e) If $\dim \mathcal{H} \leq 4$ and $W(A)$ is a polygon (with interior), then A is normal.

Compressions and dilations

Fact If A is a compression of B, equivalently, B is a dilation of A, i.e., $A = X^*BX$ for some X with $X^*X = I_{\mathcal{H}}$, equivalently, B has an operator matrix of the form $\begin{pmatrix} A & * \\ * & * \end{pmatrix}$, then

$$W(A) \subseteq W(B).$$

Question How about the converse?

Precaution What if A has dimension higher than B?

Note $W(I \otimes B) = W(B \oplus B \oplus \cdots) = W(B)$.

Refined Question

If $W(A) \subseteq W(B)$, is A has a dilation of the form $I \otimes B$?

Theorem The answer is yes in the following cases.

(a) [Ando 1973; Arveson 1972] $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

(b) [Mirman 1968; Nakamura 1982] $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$.

(c) [Choi and Li, 2001]

$B \in M_2$ or $B = \begin{pmatrix} B_1 & 0 \\ 0 & c \end{pmatrix}$ with $B_1 \in M_2$.

Examples [Choi and Li, 2000]

The answer is negative if

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

In both cases, consider $A = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$. Then ....

Refined question If $W(A) \subseteq W(B)$ and $\|A\| \leq \|B\|$, does it follow that A has a dilation of the form $I \otimes B$?

Example Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = B^t$. Then $Ae_1 = 2e_2$ and $A^2 e_1 = e_3$, but no such vector for $I \otimes B$.

Open problem If $W(A) \subseteq W(B)$ and $\|A\| \leq \|B\|$, does it follow that A has a dilation of the form $I \otimes (B \oplus B^t)$?

Additional questions

(1) Characterize / determine other B such that:
If A satisfies $W(A) \subseteq W(B)$, then A has a dilation of the form $I \otimes B$.

(2) Characterize / determine A such that:
If B satisfies $W(A) \subseteq W(B)$, then A has a dilation of the form $I \otimes B$.

(3) Determine the conditions on a pair of operators A and B such that $W(A) \subseteq W(B)$ will imply A has a dilation of the from $I \otimes B$.

More on dilations

(a) Every $A \in \mathcal{B}(\mathcal{H})$ has a normal dilation preserving norm:

$$\begin{pmatrix} A & \sqrt{\|A\|^2 I - AA^*} \\ \sqrt{\|A\|^2 I - A^* A} & -A^* \end{pmatrix}.$$

(b) [Halmos 1964] If $A \in \mathcal{B}(\mathcal{H})$, then $\overline{W(A)}$ is the intersection of all $\overline{W(N)}$ such that N is a normal dilation of A.

Question Can one remove the closure sign?

Answer [Durzt, 1964] No.

Question [Halmos] Every contraction $A \in \mathcal{B}(\mathcal{H})$ has many unitary dilations. Is it true that $\overline{W(A)}$ is the intersection of all $\overline{W(U)}$ such that U is a unitary dilation of A?

Answer [Choi and Li, 2001] The answer is affirmative because of the following result.

Theorem Suppose A is a contraction with $A + A^* \leq \mu I$. Then A has a unitary dilation U of A such that $U + U^* \leq \mu I$.

This result is equivalent to the following theorem by the duality of positive linear maps.

Theorem If $B = B_1 \oplus [c]$ with $B_1 \in M_2$, and $W(A) \subseteq W(B)$ then A has a dilation of the form $I \otimes B$.

Theorem [Choi and Li, 2001]

Consider $\phi : \text{span}\{I, B, B^*\} \to \text{span}\{I, A, A^*\}$ defined by

$$\phi(aI + bB + cB^*) = aI + bA + cA^*.$$

Then (a) $\iff$ (b) $\Rightarrow$ (c) $\iff$ (d).

(a) ϕ is completely positive, i.e., $I_m \otimes \phi$ is positive linear for all m.

(b) A has a dilation of the form $I \otimes B$.

(c) $W(A) \subseteq W(B)$.

(d) ϕ is a positive linear map.

Norm estimations

Theorem

(a) If $W(A)$ lies in the unit circle, then $\|A\| \leq 2$.

(b) If $W(A)$ lies in the triangle with vertices a, b, c, then

$$\|A\| \leq \max\{|a|, |b|, |c|\}.$$

The bounds in (a) and (b) are attainable.

Theorem [Choi and Li, 2003]
Suppose $W(A)$ lies in the rectangle

$$R = \{a + ib : a \in [a_1, a_2] \text{ and } b \in [b_1, b_2]\}$$

such that $a_2 \geq |a_1|$ and $b_2 \geq |b_1|$.

(i) If $a_1 b_2 + a_2 b_1 \geq 0$ then $\|A\| \leq |a_2 + ib_2|$.

(ii) If $a_1 b_2 + a_2 b_1 \leq 0$ then $\|A\| \leq \tau + \tau'$,

where $\tau = |a_1 - \mu_0| = |a_2 - \mu_0|$ and $\tau' = |ib_1 - \mu_0| = |ib_2 - \mu_0|$ with $\mu_0 = [(a_1 + a_2) + i(b_1 + b_2)]/2$.

The bounds in (i) and (ii) are attainable.

Remarks

(1) In the [CL] theorem, the condition means $A = H + iG$ such that $a_1 I \leq H \leq a_2 I$ and $b_1 I \leq G \leq b_2 I$.

(2) In fact, (i) happens if and only if R is contained in a triangle inside the circle with radius $|a_2 + ib_2|$. We can use the triangle result in this case. Otherwise, we use the bound $\tau + \tau'$.

Example If $W(A) \subseteq R = \{a + ib : -1 \leq a, b \leq 1\}$, then $\|A\| \leq 2$. The equality holds if $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

(3) [Choi and Li, unpublished.] We can extend the result to a parallelogram $P = L_1 + L_2$, where L_1 is a line joining α_1, α_2, and L_2 is a line joining β_1, β_2, so that $\alpha_i + \beta_j$ are the vertices of P. If P lies in a triangle in a circle with radius $|v|$ where v is a vertex of P, then

$$\|A\| \leq |v|.$$

Otherwise,

$$\|A\| \leq \tau + \tau',$$

where $\tau = |\alpha_1 - \mu_0| = |\alpha_2 - \mu_0|$, $\tau' = |\beta_1 - \mu_0| = |\beta_2 - \mu_0|$. Again, these bounds are best possible.

Remark Suppose $A = A_1 + A_2$ such that $W(A_1) \subseteq L_1$ and $W(A_2) \subseteq L_2$. Then

$$\|A\| = \|A_1 + A_2\| \leq \inf\{\|A_1 + \mu I\| + \|A_2 - \mu I\| : \mu \in \mathbb{C}\}.$$

Theorem [Choi and Li, 2006] For any $A, B \in \mathcal{B}(\mathcal{H})$,

$$\sup_{U \text{ unitary}} \|A + UBU^*\| = \inf_{\mu \in \mathbb{C}} \{\|A + \mu I\| + \|B - \mu I\|\}, \quad (\dagger)$$

and they are the same as:

$$\sup\{\|AX + XB\| : X \in \mathcal{B}(\mathcal{H}),\ \|X\| \leq 1\},$$

which can be viewed as the norm of the derivation operator $D_{A,B}$ defined by $X \mapsto AX + XB$.

Implications to spectral sets, unitary similarity orbits, and the theory of unitarily invariant norms.

Theorem Let $A, B \in \mathcal{B}(\mathcal{H})$, and let μ_0 be such that

$$\|A-\mu_0 I\| + \|B-\mu_0 I\| \le \|A-\mu I\| + \|B-\mu I\| \quad \text{for all } \mu \in \mathbb{C}.$$

If $a = \|A-\mu_0 I\|$ and $b = \|B-\mu_0 I\|$, then for any unitary U, V and polynomials $f(z), g(z)$,

$$\|U^* f(A)U + V^* g(B)V\| \le \max_{|z-\mu_0|=a} |f(z)| + \max_{|z-\mu_0|=b} |g(z)|.$$

Theorem Let $\|\cdot\|$ be a norm on M_n such that $\|UAV\| = \|A\|$ for all $A \in M_n$ and unitary $U, V \in M_n$. Then (†) holds if and only if $\|\cdot\|$ is a multiple of the operator norm.

Remark Note that $\|A\|$ is basically determined by a norm attaining unit vector x and Ax. Suppose $W(A)$ is a subseteq of a given compact subset S of $\mathbb{C}$. Then $\|A\| \leq \|B\|$ for some $B \in M_2$ such that $W(B) \subseteq S$. Thus, the optimal bound for $\|A\|$ equals

$$\gamma(S) = \max\{\|B\| : B \in M_2, \ W(B) \subseteq S\}.$$

Question Can we have a good formula for $\gamma(S)$?

Thank you for your attention!