Numerical ranges, dilations, and norm estimations

Chi-Kwong Li
Department of Mathematics
College of William and Mary
Williamsburg, Virginia 23187-8795

Mainly from joint work with:
Man-Duen Choi (University of Toronto)
Notation and definitions

Let $\mathcal{H}$ be a Hilbert space.

Let $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$.

If $\mathcal{H}$ has dimension $n$, identify $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ with $(\mathbb{C}^n, M_n)$.

The numerical range of $A \in \mathcal{B}(\mathcal{H})$ is:

$$W(A) = \{(Ax, x) : x \in \mathcal{H}, \langle x, x \rangle = 1\}.$$

Examples

If $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $W(A) = [0, 1]$.

If $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ then $W(A)$ is the unit disk.
Basic properties

(a) $W(U^*AU) = W(A)$ for any unitary $U \in \mathcal{B}(\mathcal{H})$.
(b) $W(rA + sI) = rW(A) + s$.
(c) $W(A + E) \subseteq W(A) + W(E)$.
(d) $\sigma(A) \subseteq W(A)$.

By (c) and (d), we have

$$\sigma(A + E) \subseteq W(A + E) \subseteq W(A) + W(E).$$

In general, $\sigma(A + E) \nsubseteq \sigma(A) + \sigma(E)$.

Example Let $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$. Then

$\sigma(A) = \sigma(E) = \{0\}$ and $\sigma(A + E) = \left\{ \pm \sqrt{M\epsilon} \right\} \subseteq W(A + E)$.
Some geometrical/topological properties
(a) \( W(A) \) is bounded.
(b) \( W(A) \) is closed if \( \dim \mathcal{H} < \infty \).
(c) [Toeplitz-Hausdorff, 1918-19] \( W(A) \) is always convex.
(d) If \( A = U^* \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix} U \) then \( W(A) \) is an elliptical disk with foci \( \lambda_1, \lambda_2 \), and minor axis with length \( |b| \).

Remark To prove (d), Li [1996] showed that one needs only to verify the (easy) cases when
\[
A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}
\]
because \( W(f(A)) = f(W(A)) \) for any real affine transform \( f \) on \( \mathcal{C} \) (identified as \( \mathbb{R}^2 \)).
More useful properties

(a) \( A = A^* \) if and only if \( W(A) \subseteq \mathbb{R} \).

(b) \( A = A^* \) is positive semi-definite if and only if

\[
W(A) \subseteq [0, \infty).
\]

(c) \( A = \lambda I \) if and only if \( W(A) = \{ \lambda \} \).

(d) If \( A \) is normal, then \( \overline{W(A)} = \text{conv} \sigma(A) \). Furthermore, if \( \sigma(A) \) is finite, then \( W(A) \) is a convex polygon.

(e) If \( \dim \mathcal{H} \leq 4 \) and \( W(A) \) is a polygon (with interior), then \( A \) is normal.
Compressions and dilations

**Fact** If $A$ is a compression of $B$, equivalently, $B$ is a dilation of $A$, i.e., $A = X^* B X$ for some $X$ with $X^* X = I_H$, equivalently, $B$ has an operator matrix of the form \[
\begin{pmatrix}
A & * \\
* & *
\end{pmatrix},
\]
then \[W(A) \subseteq W(B).\]

**Question** How about the converse?

**Precaution** What if $A$ has dimension higher than $B$?

**Note** $W(I \otimes B) = W(B \oplus B \oplus \cdots) = W(B)$. 

Refined Question

If $W(A) \subseteq W(B)$, is $A$ has a dilation of the form $I \otimes B$?

Theorem The answer is yes in the following cases.

(a) [Ando 1973; Arveson 1972] $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

(b) [Mirman 1968; Nakamura 1982] $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$.

(c) [Choi and Li, 2001] $B \in M_2$ or $B = \begin{pmatrix} B_1 & 0 \\ 0 & c \end{pmatrix}$ with $B_1 \in M_2$. 
Examples [Choi and Li, 2000]

The answer is negative if

\[
B = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\quad \text{or} \quad
B = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}.
\]

In both cases, consider \( A = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \). Then ...
Refined question If \( W(A) \subseteq W(B) \) and \( \|A\| \leq \|B\| \), does it follow that \( A \) has a dilation of the form \( I \otimes B \)?

Example Let \( A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = B^t \). Then \( Ae_1 = 2e_2 \) and \\
\( A^2e_1 = e_3 \), but no such vector for \( I \otimes B \).

Open problem If \( W(A) \subseteq W(B) \) and \( \|A\| \leq \|B\| \), does it follow that \( A \) has a dilation of the form \( I \otimes (B \oplus B^t) \)?
Additional questions

(1) Characterize / determine other $B$ such that:
   If $A$ satisfies $W(A) \subseteq W(B)$, then $A$ has a dilation of the form $I \otimes B$.

(2) Characterize / determine $A$ such that:
   If $B$ satisfies $W(A) \subseteq W(B)$, then $A$ has a dilation of the form $I \otimes B$.

(3) Determine the conditions on a pair of operators $A$ and $B$ such that $W(A) \subseteq W(B)$ will imply $A$ has a dilation of the form $I \otimes B$. 
More on dilations

(a) Every $A \in \mathcal{B}(\mathcal{H})$ has a normal dilation preserving norm:

$$
\begin{pmatrix}
A \\
\sqrt{\|A\|^2 I - AA^*}
\end{pmatrix}
\begin{pmatrix}
\sqrt{\|A\|^2 I - AA^*} \\
-A^*
\end{pmatrix}.
$$

(b) [Halmos 1964] If $A \in \mathcal{B}(\mathcal{H})$, then $\overline{W(A)}$ is the intersection of all $\overline{W(N)}$ such that $N$ is a normal dilation of $A$.

**Question** Can one remove the closure sign?

**Answer** [Durzt, 1964] No.
**Question** [Halmos] Every contraction $A \in B(\mathcal{H})$ has many unitary dilations. Is it true that $\overline{W(A)}$ is the intersection of all $\overline{W(U)}$ such that $U$ is a unitary dilation of $A$?

**Answer** [Choi and Li, 2001] The answer is affirmative because of the following result.

**Theorem** Suppose $A$ is a contraction with $A + A^* \leq \mu I$. Then $A$ has a unitary dilation $U$ of $A$ such that $U + U^* \leq \mu I$.

This result is equivalent to the following theorem by the duality of positive linear maps.

**Theorem** If $B = B_1 \oplus [c]$ with $B_1 \in M_2$, and $W(A) \subseteq W(B)$ then $A$ has a dilation of the form $I \otimes B$. 
Theorem [Choi and Li, 2001]

Consider $\phi : \text{span} \{I, B, B^*\} \rightarrow \text{span} \{I, A, A^*\}$ defined by

$$\phi(aI + bB + cB^*) = aI + bA + cA^*.$$ 

Then (a) $\iff$ (b) $\implies$ (c) $\iff$ (d).

(a) $\phi$ is completely positive, i.e., $I_m \otimes \phi$ is positive linear for all $m$.

(b) $A$ has a dilation of the form $I \otimes B$.

(c) $W(A) \subseteq W(B)$.

(d) $\phi$ is a positive linear map.
Norm estimations

Theorem

(a) If $W(A)$ lies in the unit circle, then $\|A\| \leq 2$.

(b) If $W(A)$ lies in the triangle with vertices $a, b, c$, then

$$\|A\| \leq \max\{|a|, |b|, |c|\}.$$ 

The bounds in (a) and (b) are attainable.
**Theorem** [Choi and Li, 2003]

Suppose $W(A)$ lies in the rectangle

$$R = \{a + ib : a \in [a_1, a_2] \text{ and } b \in [b_1, b_2]\}$$

such that $a_2 \geq |a_1|$ and $b_2 \geq |b_1|$.

(i) If $a_1 b_2 + a_2 b_1 \geq 0$ then $\|A\| \leq |a_2 + ib_2|$.

(ii) If $a_1 b_2 + a_2 b_1 \leq 0$ then $\|A\| \leq \tau + \tau'$,

where $\tau = |a_1 - \mu_0| = |a_2 - \mu_0|$ and $\tau' = |ib_1 - \mu_0| = |ib_2 - \mu_0|$

with $\mu_0 = [(a_1 + a_2) + i(b_1 + b_2)]/2$.

The bounds in (i) and (ii) are attainable.
Remarks
(1) In the [CL] theorem, the condition means $A = H + iG$
such that $a_1 I \leq H \leq a_2 I$ and $b_1 I \leq G \leq b_2 I$.

(2) In fact, (i) happens if and only if $R$ is contained in a triangle inside the circle with radius $|a_2 + ib_2|$. We can use the triangle result in this case. Otherwise, we use the bound $\tau + \tau'$.

Example If $W(A) \subseteq R = \{a + ib : -1 \leq a, b \leq 1\}$, then $\|A\| \leq 2$. The equality holds if $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. 
(3) [Choi and Li, unpublished.] We can extend the result to a parallelogram $P = L_1 + L_2$, where $L_1$ is a line joining $\alpha_1, \alpha_2$, and $L_2$ is a line joining $\beta_1, \beta_2$, so that $\alpha_i + \beta_j$ are the vertices of $P$. If $P$ lies in a triangle in a circle with radius $|v|$ where $v$ is a vertex of $P$, then

$$\|A\| \leq |v|.$$ 

Otherwise,

$$\|A\| \leq \tau + \tau',$$

where $\tau = |\alpha_1 - \mu_0| = |\alpha_2 - \mu_0|$, $\tau' = |\beta_1 - \mu_0| = |\beta_2 - \mu_0|$.

Again, these bounds are best possible.
**Remark** Suppose $A = A_1 + A_2$ such that $W(A_1) \subseteq L_1$ and $W(A_2) \subseteq L_2$. Then

$$\|A\| = \|A_1 + A_2\| \leq \inf\{\|A_1 + \mu I\| + \|A_2 - \mu I\| : \mu \in \mathbb{C}\}.$$  

**Theorem** [Choi and Li, 2006] For any $A, B \in \mathcal{B}(\mathcal{H})$,

$$\sup_{U \text{ unitary}} \|A + UBU^*\| = \inf_{\mu \in \mathbb{C}} \{\|A + \mu I\| + \|B - \mu I\|\}, \quad (\dagger)$$

and they are the same as:

$$\sup\{\|AX + XB\| : X \in \mathcal{B}(\mathcal{H}), \|X\| \leq 1\},$$

which can be viewed as the norm of the derivation operator $D_{A,B}$ defined by $X \mapsto AX + XB$. 
Implications to spectral sets, unitary similarity orbits, and the theory of unitarily invariant norms.

**Theorem** Let \( A, B \in \mathcal{B}(\mathcal{H}) \), and let \( \mu_0 \) be such that

\[
\| A - \mu_0 I \| + \| B - \mu_0 I \| \leq \| A - \mu I \| + \| B - \mu I \| \quad \text{for all } \mu \in \mathbb{C}.
\]

If \( a = \| A - \mu_0 I \| \) and \( b = \| B - \mu_0 I \| \), then for any unitary \( U, V \) and polynomials \( f(z), g(z) \),

\[
\| U^* f(A)U + V^* g(B)V \| \leq \max_{|z-\mu_0|=a} |f(z)| + \max_{|z-\mu_0|=b} |g(z)|.
\]

**Theorem** Let \( \| \cdot \| \) be a norm on \( M_n \) such that \( \| UAV \| = \| A \| \) for all \( A \in M_n \) and unitary \( U, V \in M_n \). Then (1) holds if and only if \( \| \cdot \| \) is a multiple of the operator norm.
Remark Note that $\|A\|$ is basically determined by a norm attaining unit vector $x$ and $Ax$. Suppose $W(A)$ is a subseteq of a given compact subset $S$ of $\mathcal{C}$. Then $\|A\| \leq \|B\|$ for some $B \in M_2$ such that $W(B) \subseteq S$. Thus, the optimal bound for $\|A\|$ equals

$$\gamma(S) = \max\{\|B\| : B \in M_2, W(B) \subseteq S\}.$$ 

Question Can we have a good formula for $\gamma(S)$?
Thank you for your attention!