Distance Preserving Maps on Matrices

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General problem

Suppose $\|\cdot\|$ is a norm on a matrix space $\mathcal{M}$. Characterize $\phi : \mathcal{M} \to \mathcal{M}$ such that

(a) ϕ is linear/additive/multiplicative and

$$\|\phi(A)\| = \|A\| \text{ for all } A \in \mathcal{M}.$$

(b) ϕ satisfies $\|\phi(A) \bullet \phi(B)\| = \|A \bullet B\|$ for all $A, B \in \mathcal{M}$, where $A \bullet B$ is one of the following operations:

$$A + B,\ A - B,\ AB,\ ABA,\ (AB + BA)/2, \text{ etc.}$$

(c) If you are not interested in norm, consider other functions F and study ϕ such that $F(\phi(A) \bullet \phi(B)) = F(A \bullet B)$ for all $A, B \in \mathcal{M}$.

Unitarily invariant norms

A norm $\|\cdot\|$ on $M_{m,n}$ is *unitarily invariant* (UI) if

$$\|UAV\| = \|A\|$$

for all $A \in M_{m,n}$ and unitary matrices $U \in M_m, V \in M_n$. Evidently,

1. UI norms depend only on the singular valus of A:

$$s_1(A) \geq s_2(A) \geq \cdots.$$

2. Let U and V be unitary, and define $\phi : M_{m,n} \to M_{m,n}$ by

$$\text{(a) } A \mapsto UAV, \quad \text{or} \quad \text{(b) } m = n \text{ and } A \mapsto UA^tV.$$

   Then ϕ is a linear isometry for $\|\cdot\|$.

Theorem [Li and Tsing, 1990] Let $\|\cdot\|$ be a UI norm on $M_{m,n}$ not equal to a multiple of the Frobenius norm. A linear isometry of $\|\cdot\|$ has the form (a) or (b).

Special cases

1. [Kadison, 1951] The *operator norm* $\|A\|_{\text{op}} = s_1(A)$.

2. [Russo, 1969] The *trace norm* $\|A\|_{\text{tr}} = s_1(A) + \cdots + s_n(A)$.

3. [Arazy, 1975] For $p \geq 1$, the *Schatten p-norm*

   $$S_p(A) = (s_1(A)^p + \cdots + s_n(A)^p)^{1/p}.$$

   When $p = 2$, it reduces to the Frobenius norm.

4. [Grone & Marcus, 1976; Grone, 1977] For $1 \leq k \leq n$, the *Ky Fan k-norm* $\quad F_k(A) = s_1(A) + \cdots + s_k(A)$.

5. [Li & Tsing, 1988] For $1 \leq k \leq \min\{m, n\}$ and $p \geq 1$, the *(p, k)-norm* $\|A\|_{(p,k)} = (s_1(A)^p + \cdots + s_k(A)^p)^{1/p}$.

6. [Li & Tsing, 1988] Given $c = (c_1, \ldots, c_n)$ with $c_1 \geq \cdots \geq c_n$, the *c-norm* $\quad \|A\|_c = c_1 s_1(A) + \cdots + c_n s_n(A)$.

Some proof techniques

Geometrical approach

Let $\|\cdot\|$ be a norm on M_n. Denote the unit norm ball by

$$\mathcal{B} = \mathcal{B}_{\|\cdot\|} = \{A \in M_n : \|A\| \leq 1\},$$

and the set of its extreme points by $\mathcal{E} = \mathcal{E}_{\|\cdot\|}$.

A linear map $\phi : M_n \to M_n$ is an isometry for $\|\cdot\|$ if and only if $\phi(\mathcal{E}) = \mathcal{E}$.

Example The set of extreme points of the unit ball of the operator norm is the group $\mathcal{U}_n$ of unitary matrices.

Sometimes, special (algebraic/differential) geometrical features of boundary points were used.

Duality technique

Let $(A, B) = \mathrm{tr}\,(AB^*) = \sum a_{ij}\bar{b}_{ij}$ be the inner product on $M_{m,n}$. Define the *dual norm* of $\|\cdot\|$ on M_n by

$$\|A\|^* = \max\{|(A, B)| : \|B\| \leq 1\},$$

and the *dual transformation* of a linear map $\phi : M_n \to M_n$ to be the unique linear map $\phi^* : M_n \to M_n$ such that

$$(\phi(A), B) = (A, \phi^*(B)) \quad \text{for all } A, B \in M_n.$$

Facts

(a) If ϕ is an isometry for $\|\cdot\|$, then ϕ^* is an isometry for $\|\cdot\|^*$.

(b) If ϕ has the standard form, then so is ϕ^*.

(c) The operator norm and the trace norm are dual to each other.

A group theory approach

The set of linear isometries for a norm form a group. Let $\mathcal{U}_m \times \mathcal{U}_n$ be the group of operators of the form $A \mapsto UAV$ for a pair of unitary matrices U, V.

Theorem [Djokovic & Li, 1994] Let G be the isometry group of a given UI norm on $m \times n$ complex matrices. Then one of the following holds.

(a) $\mathcal{U}_{mn}$.

(b) $m \neq n$ and $G = \mathcal{U}_m \times \mathcal{U}_n$.

(c) $m = n$ and $G = \langle \mathcal{U}_m \times \mathcal{U}_n, \tau \rangle$, where $\tau(A) = A^t$.

The real case

Theorem [Li & Tsing, 1990], [Djokovic & Li, 1994]. Let G be the isometry group of a UI norm on $m \times n$ real matrices. Then one of the following holds.

(a) $G = \mathcal{U}_{mn}$.

(b) $m \neq n$ and $G = \mathcal{U}_m \times \mathcal{U}_n$.

(c) $m = n$ and $G = \langle \mathcal{U}_m \times \mathcal{U}_n, \tau \rangle$.

(d) $m = n = 4$ and $G = \langle \mathcal{U}_4 \times \mathcal{U}_4, \tau, \psi \rangle$,

$$\psi(A) = (A + B_1 A C_1 + B_2 A C_2 + B_3 A C_3)/2 \qquad \text{with}$$

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Facts about ψ

[Johnson, Laffey, Li, 1988], [Djokovic, 1990], [Li & Tsing, 1990], [Chang & Li, 1991], [Djokovic & Li, 1994].

1. $\{I, B_1, B_2, B_3\}$ and $\{I, C_1, C_2, C_3\}$ are two basis for real quaternions in 4×4 matrices.

2. There is a 4-dimensional subspace $\mathbf{W}$ of M_4 such that

$$\psi(A_1 + A_2) = A_1 - A_2 \qquad A_1 \in \mathbf{W}^\perp,\ A_2 \in \mathbf{W}.$$

3. If A has singular values $s_1 \geq s_2 \geq s_2 \geq s_4$, then $\psi(A)$ has singular values

$$(s_1 + s_2 + s_3 + \delta_A s_4)/2, \quad (s_1 + s_2 - s_3 - \delta_A s_4)/2,$$
$$(s_1 - s_2 + s_3 - \delta_A s_4)/2, \quad (s_1 - s_2 - s_3 - \delta_A s_4)/2,$$

   where δ_A is the sign of $\det(A)$. So, ψ preserves the Ky Fan 2-norm.

4. The set of extreme point of the unit norm ball of the Fy Fan 2-norm on $n \times n$ real matrices have three connected components:

$$\frac{1}{2}\mathcal{O}_n^+ = \frac{1}{2}\{P : P^t P = I_n,\ \det(P) = 1\},$$

$$\frac{1}{2}\mathcal{O}_n^- = \frac{1}{2}\{P : P^t P = I_n,\ \det(P) = -1\},$$

$$\mathcal{R} = \{xy^t : x, y \in \mathbb{R}^n,\ l_2(x) = l_2(y) = 1\}.$$

If $n \neq 4$, $\phi(\mathcal{R}) = \mathcal{R}$; if $n = 4$, the special map ψ permutes these components.

General distance preserving maps

Characterize mappings $\phi : M_{m,n} \to M_{m,n}$ such that

$$\|\phi(A) - \phi(B)\| = \|A - B\| \qquad \text{for all } A, B \in M_{m,n}.$$

By a result of Charzyński [1953], such a map is real affine.

Theorem [Chan, Li, Sze, 2005] In the complex case, ϕ has the form

$$A \mapsto UAV + R, \quad A \mapsto U\overline{A}V + R,$$

or when $m = n$

$$A \mapsto UA^tV + R, \qquad \text{or} \qquad A \mapsto UA^*V + R.$$

A more natural question

Characterize mappings $\phi : M_{m,n} \to M_{m,n}$ such that

$$\|\phi(A) + \phi(B)\| = \|A + B\| \qquad \text{for all } A, B \in M_{m,n}.$$

Note that

$$\|\phi(A) + \phi(-A)\| = \|A - A\| = 0.$$

So, $\phi(-A) = -\phi(A)$ and

$$\|\phi(A) - \phi(B)\| = \|\phi(A) + \phi(-B)\| = \|A - B\|.$$

Consequently, ϕ is real linear.

Multiplicative maps and additional results

[Cheung, Fallat, Li, 2002; Guralnick, Li, Rodman, 2003]

Theorem Let $\|\cdot\|$ be a USI norm, and let $\mathbf{V} = M_n^{(k)}$, SL_n, or GL_n. A mapping $\phi : \mathbf{V} \to M_n$ is multiplicative and satisfies

$$\|\phi(A)\| = \|A\| \text{ for all } A \in M_n$$

if and only if ϕ has the form

$$A \mapsto f(\det(A))U^*AU \quad \text{ or } \quad A \mapsto f(\det(A))U^*\overline{A}U.$$

Key step Apply results on multiplicative maps on $M_n^{(k)}$, SL_n, GL_n by [Jodeit and Lam, 1969] and [Borel and Tits, 1978].

Theorem [Chan, Li, Sze, 2005] Let $\|\cdot\|$ be a USI norm. A mapping $\phi : M_n^{(k)} \to M_n$ satisfies

$$\|AB\| = \|\phi(A)\phi(B)\| \text{ for all } A, B \in M_n$$

has the form

$$A \mapsto U^* h(A) U \qquad \text{or} \qquad A \mapsto U^* h(\overline{A}) U,$$

where

$$h(X) = U_X X = X V_X \text{ for some unitary } U_X, V_X.$$

Suppose $\|xy^t\|$ is not always a constant multiple of $s_1(xy^t)$. Then ϕ has the form

$$A \mapsto \mu_A U^* A U \qquad \text{or} \qquad A \mapsto \mu_A U^* \overline{A} U, \qquad\qquad |\mu_A| = 1.$$

Key step Determine mappings such that

$$\phi(A)\phi(B) = 0 \text{ whenever } AB = 0.$$

Further research

Characterize $\phi : \mathbf{V} \to \mathbf{V}'$ such that

$$\|\phi(A) \bullet \phi(B)\| = \|A \bullet B\| \qquad \text{for all } A, B \in \mathbf{V},$$

where $\bullet$ is a certain operation on matrices such as

$A \pm B$, AB, AB^{-1}, AB^*,

the Jordan product $(AB + BA)/2$,

the Lie product $AB - BA$,

the Jordan triple product ABA, etc.

More generally, let F be a norm or other functions on $\mathbf{V}$ such as the spectrum, *spectral radius*, rank, numerical range, etc.

Characterize $\phi : \mathbf{V} \to \mathbf{V}'$ such that

$$F(\phi(A) \bullet \phi(B)) = F(A \bullet B) \qquad \text{for all } A, B \in \mathbf{V}.$$

What about other domains $\mathbf{V}$ and co-domains $\mathbf{V}'$?

* [Li, Šemrl, Sourour, 2003] surjective isometries for Ky Fan k-norm between nest algebras in M_n.
* [Cheung, Li, Poon, 2004] (non-surjective) isometries for the operator norm between rectangular matrix spaces.
* [Li, Poon, Sze, 2005] (non-surjective) isometries for the Ky Fan norms between rectangular matrix spaces.
* [Sourour, 1981] surjective isometries for UI norms on symmetrically normed ideals.
* [Anoussis and Katavolos, 1995] surjective isometries for Schatten p-norms on nest algebras.
* [Arazy, Solel, Moore, Trent, etc. 1980-present] surjective isometries for the operator norm on non-self-adjoint algebras, nest algebras, reflexive operator algebras, CSL algebras, etc.

Comments, answers, are more questions are welcome!

Thank you for your attention!