H-unitary and Lorentz matrices

Chi-Kwong Li
Department of Mathematics
College of William and Mary
Williamsburg, VA 23185

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Yik-Hoi Au-Yeung (HKU) and Leiba Rodman (W&M)

and an LAA paper with

Nung-Sing Sze (HKU)

1. Introduction

$M_n = M_n(\mathbb{F})$: $n \times n$ matrices over $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$.

Let $H^* = H \in M_n$ be invertible.

Define an indefinite inner product on $\mathbb{F}^n$ by

$$[x, y]_H = [x, y] = y^* H x.$$

A matrix $A \in M_n$ is called H-*unitary* if

$$[Ax, Ay] = [x, y] \text{ for all } x, y \in \mathbb{F}^n,$$

equivalently,

$$A^* H A = H.$$

When $H = I$ we get the usual inner product and the unitary matrices.

Proposition If $S \in M_n$ is invertible, then $A \in M_n$ is H-unitary if and only if $S^{-1}AS$ is S^*HS-unitary. Thus, we can often consider the case $H = J = J_{p,q} = I_p \oplus -I_q$.

Notation

$\mathcal{U}(H)$: the group of H-unitary matrices.

$\mathcal{U}_{p,q}$: the group of unitary matrices of the form $U_1 \oplus U_2$ with $U_1 \in M_p$ and $U_2 \in M_q$.

Note that $\mathcal{U}_{p,q}$ is a subgroup of $\mathcal{U}(J_{p,q})$.

Some Motivations

* Group Theory
* Physics - Lorentz matrices: $A^t H A = H$ for $H = I_3 \oplus [-1]$
* Numerical analysis, differential equations, electrical engineering - functions with values in $\mathcal{U}(H)$.

Our goal

* Give a gentle introduction via the canonical form, and
* discuss some recent results on linear preservers.

Some related recent work

* Higham, Horn, Merino, Edward Poon have obtained some interesting results recently.

* Bebiano, Providencia, Rodman, Tsing, Uhlig, Li, etc. have studied the subject via H-numerical ranges.

2. CS Decomposition

CS (cos − sin) decomposition of unitary matrices

For any unitary matrix A there are matrices $X, Y \in \mathcal{U}_{p,q}$ such that

$$XAY = I_r \oplus \begin{pmatrix} C & S \\ -S^t & C \end{pmatrix} \oplus I_s,$$

where $C, S \in M_{p-r}$ are diagonal matrices with positive diagonal entries satisfying $C^2 + S^2 = I_{p-r}$.

CS $(\cosh - \sinh)$ **decomposition for** J**-unitary matrices**

Theorem 1 A matrix $A \in \mathcal{U}(J_{p,q})$ if and only if there exist $X, Y \in \mathcal{U}_{p,q}$ such that

$$XAY = I_r \oplus \begin{pmatrix} \sqrt{I_{p-r} + D^2} & D \\ D & \sqrt{I_{p-r} + D^2} \end{pmatrix} \oplus I_s,$$

where $D \in M_{p-r}$ is a diagonal matrices with positive diagonal entries $d_1, \ldots, d_r$ determined by the singular values of A:

$$\sqrt{1 + d_1^2} \pm d_1, \ldots, \sqrt{1 + d_r^2} \pm d_r, \underbrace{1, \ldots, 1}_{n-2r}.$$

Proof. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \text{with } A_{11} \in M_p.$$

Suppose $U \in M_p$ and $V \in M_q$ are such that A_{12} has singular value decomposition UDV. Let

$$X = U \oplus I_q \qquad \text{and} \qquad Y = I_p \oplus V.$$

Then with some adjustments of X and Y, XAY has the desired form. ■

Theorem 2 Let $A \in M_n$. Then $A \in \mathcal{U}(J_{p,q})$ if and only if there is $U \in \mathcal{U}_{p,q}$ such that UA has any one of the following forms:

(a) $\begin{pmatrix} \sqrt{I_p + MM^*} & M \\ M^* & \sqrt{I_q + M^*M} \end{pmatrix}$ for a $p \times q$ matrix M,

(b) $\exp\left(\begin{pmatrix} 0_p & L \\ L^* & 0_q \end{pmatrix}\right)$ for a $p \times q$ matrix L.

(c) $\begin{pmatrix} I_p & K \\ K^* & I_q \end{pmatrix}\begin{pmatrix} I_p & -K \\ -K^* & I_q \end{pmatrix}^{-1}$ for a $p \times q$ matrix K

with all singular values less than one.

Proof. By Theorem 1, XAY is a direct sum of 2×2 matrices. So, the problem reduces to 2×2 matrix computations. ■

Remarks

1. One can write analogous conditions AV with $V \in J_{p,q}$. We omit the statements.

2. Notice that the matrices in (a), (b), and (c) are just the positive definite part of the polar decomposition of A. Hence, they are the same as $(A^*A)^{1/2}$.

Corollary Suppose $A \in \mathcal{U}(J_{p,q})$ and $U = U_1 \oplus U_2 \in \mathcal{U}_{p,q}$ are such that

$$UA = \begin{pmatrix} I_p & K \\ K^* & I_q \end{pmatrix} \begin{pmatrix} I_p & -K \\ -K^* & I_q \end{pmatrix}^{-1}.$$

as in Theorem 2 (c). Let $(X_0 \,|\, Y_0), (X_1 \,|\, Y_1) \in M_{r\times p} \times M_{r\times q}$ be *initial vectors* and *final vectors* of A, respectively, i.e.,

$$(X_0 \,|\, Y_0)A = (X_1 \,|\, Y_1).$$

Then

$$X_0 U_1^*(I_p - KK^*) = X_1(I_p + KK^*) - 2Y_1K^*, \text{ and}$$

$$Y_0 U_2^*(I_q - K^*K) = Y_1(I_q + K^*K) - 2X_1K.$$

In particular, If $X_0 \in M_p(\mathbb{F})$ is invertible, then

$$X_0^{-1}[X_1(I_p + KK^*) - 2Y_1K^*]$$

is a constant matrix, that is, independent of $(X_0 \,|\, Y_0)$ and of $(X_1 \,|\, Y_1)$. Similarly, if $Y_0 \in M_q(\mathbb{F})$ is invertible, then

$$Y_0^{-1}[Y_1(I_q + K^*K) - 2X_1K]$$

is a constant matrix.

Suppose $\mathbb{F} = \mathbb{R}$ and $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{U}(J_{p,q})$. Define

$$\sigma_+(A) = \mathrm{sign}(\det(A_{11})) \text{ and } \sigma_-(A) = \mathrm{sign}(\det(A_{22})).$$

Corollary Suppose $\mathbb{F} = \mathbb{R}$ and $A, B \in \mathcal{U}(J_{p,q})$. Then

$$\sigma_+(A)\sigma_+(B) = \sigma_+(AB) \qquad \sigma_-(A)\sigma_-(B) = \sigma_-(AB).$$

Corollary Suppose H is congruent to $J_{p,q}$. Then $\mathcal{U}(H)$ is homeomorphic to $\mathcal{U}_p \times \mathcal{U}_q \times \mathbb{F}^{pq}$.

If $\mathbb{F} = \mathbb{R}$, $\mathcal{U}(H)$ is a real analytic manifold consisting of four arc-wise connected components, and the identity component is locally isomorphic to

$$\mathbb{R}^{p(p-1)/2} \times \mathbb{R}^{q(q-1)/2} \times \mathbb{R}^{pq}.$$

If $\mathbb{F} = \mathbb{C}$, $\mathcal{U}(H)$ is a real analytic manifold consisting of one arc-wise connected component which is locally isomorphic to

$$\mathbb{R}^{p^2} \times \mathbb{R}^{q^2} \times \mathbb{R}^{2pq}.$$

Corollary $\mathcal{U}_{p,q}$ is a maximal bounded subgroup of $\mathcal{U}(J_{p,q})$.

Theorem 3 Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ with $A_{11} \in M_p$ and $A_{22} \in M_q$. Then $A \in \mathcal{U}(J_{p,q})$ if and only if

$$X = A_{11}(I_p + A_{21}^* A_{21})^{-1/2} \in \mathcal{U}_p,$$

$$Y = A_{22}(I_q + A_{12}^* A_{12})^{-1/2} \in \mathcal{U}_q,$$

$$\text{and} \qquad X^* A_{12} = A_{21}^* Y.$$

3. Some Factorization Results

3.1 Products of positive definite $J_{p,q}$-unitary matrices

Fact A matrix $A \in M_n(\mathbb{F})$ is a product of two positive definite matrices if and only if A is diagonalizable and has positive eigenvalues.

Proof. If $A = S^{-1}DS$, then $A = S^{-1}(S^{-1})^*S^*DS$. For the converse, if $A = PQ$ for some positive definite matrices P and Q, then A is similar to $Q^{1/2}PQ^{1/2}$

Deeper result Every $A \in M_n$ with positive determinant is a product of no more than five positive definite matrices.

Theorem 4 Let $A \in \mathcal{U}(J_{p,q})$. The following are equivalent.

(a) A is the product of two matrices of the form

$$\begin{pmatrix} \sqrt{I_p + XX^*} & X \\ X^* & \sqrt{I_q + X^*X} \end{pmatrix}, \quad X \in M_{p\times q}(\mathbb{F}).$$

(b) A is $J_{p,q}$-unitarily similar to a matrix of the form

$$\begin{pmatrix} \sqrt{I_p + XX^*} & X \\ X^* & \sqrt{I_q + X^*X} \end{pmatrix}, \quad X \in M_{p\times q}(\mathbb{F}).$$

(c) A has positive eigenvalues and is diagonalizable.

3.2 Products of reflections / Householder transforms

In $M_n(\mathbb{R})$, a matrix of the form $T_v = I - 2vv^t$ is called a reflection, and $T_v(x) = x - 2(v^t x)v$ is just a reflection of the vector x about the plane $v^\perp$.

If v is in the linear span of two basic vectors e_i and e_j, then we say that T_v is an *elementary reflection* or *Householder transform.* Thus, an elementary reflection is a direct sum of a two by two matrix and I_{n-2}.

Fact Every orthogonal/unitary matrix is the product of no more than $n(n-1)/2 + 1$ elementary reflections.

For the indefinite inner product $[x,y]_H = y^t Hx$, define a *reflection* by $T_v = I - 2vv^tH/(v^tHv)$ for $v \in \mathbb{F}^n$ with $v^tHv \neq 0$. Then T_v is an H-unitary matrix such that

$$T_v(x) = x - 2[x,v]v/[v,v].$$

Similarly, one can define *elementary reflections.*

Theorem 6 A matrix $A \in M_n(\mathbb{R})$ is $J_{p,q}$-unitary if and only if it is a product of at most

$$f(p,q) = p(p-1) + q(q-1) + \min\{p,q\} + 4$$

so many elementary reflections.

Remark Using (general) reflections instead of elementary reflections, one can express a matrix in $\mathcal{U}(p,q)$ as a product of no more than n reflections, a result due to Cartan.

4. Other canonical forms

In connection to stability theory or differential equations, one needs the canonical form in terms of the Jordan forms of A. To get this, we use the idea of linear fractional transforms.

A matrix $K \in M_n$ is called *H-skewadjoint* if

$$[Kx, y] = -[x, Ky] \text{ for every } x, y, \in \mathbb{F}^n,$$

$$\text{i.e., } HKH^{-1} = -K^*.$$

Assume that $H \in M_n$ is an invertible Hermitian (symmetric in the real case) matrix.

Proposition Suppose A is H-unitary, and $\mu, \xi \in \mathbb{F}$ satisfy $|\mu| = 1$ with $\det(A - \mu I) \neq 0$ and $-\bar{\xi} \neq \xi$ if $\mathbb{F} = \mathbb{C}$. Then the operator

$$K = (\xi A + \mu\bar{\xi} I)(A - \mu I)^{-1}$$

is H-skewadjoint such that $\det(K - \xi I) \neq 0$.

Conversely, suppose $K \in M_n$ is H-skewadjoint, and $\mu, \xi \in \mathbb{F}$ satisfy $|\mu| = 1$, $\det(K - \xi I) \neq 0$, and $-\bar{\xi} \neq \xi$ if $\mathbb{F} = \mathbb{C}$. Then

$$A = \mu(K + \bar{\xi} I)(K - \xi I)^{-1}$$

is H-unitary such that $\det(A - \mu I) \neq 0$.

5. Linear preservers

One often considers linear preservers of functions, relations, and subsets.

* [Frobenius, 1879] A linear operator $\phi : M_n \to M_n$ satisfies $\det(A) = \det(\phi(A))$ for all $A \in M_n$ if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that ϕ has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN.$$

* [Hua, 1951] A transformation $\phi : M_n \to M_n$ satisfies $\operatorname{rank}(\phi(A) - \phi(B)) = 1$ whenever $\operatorname{rank}(A - B) = 1$ if and only if there are $M, N, R \in M_n$ with $\det(MN) \neq 0$ such that ϕ has the form

$$A \mapsto MAN + R \quad \text{or} \quad A \mapsto MA^tN + R.$$

* [Marcus, 1959] A linear operator $\phi : M_n \to M_n$ satisfies $\phi(\mathcal{U}(I_n)) \subseteq \mathcal{U}(I_n)$ if and only if there are unitary matrices $U, V \in M_n$ such that ϕ has the form

$$A \mapsto UAV \quad \text{or} \quad A \mapsto UA^tV.$$

* [Cheung & Li, 2002] There is a linear operator $\phi : M_n \to M_m$ such that $\phi(\mathcal{U}(I_n)) \subseteq \mathcal{U}(I_m)$ if and only if $m = kn$ and there are unitary matrices U and V such that

$$A \mapsto U[(I_{k_1} \otimes A) \oplus (I_{k_2} \otimes A^t)]V.$$

Theorem 7 Let $H_1 \in M_n$ and $H_2 \in M_m$ be invertible Hermitian matrices. There is a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathcal{U}(H_1)) \subseteq \mathcal{U}(H_2)$ if and only if m is a multiple of n, and there exist invertible matrices $S \in M_m$, $U, V \in \mathcal{U}(H_2)$ such that

$$S^* H_2 S = [(I_a \oplus -I_b) \otimes H_1] \oplus [(I_c \oplus -I_d) \otimes (H_1^{-1})^t],$$

and ϕ has the form

$$A \mapsto US[(I_{a+b} \otimes A) \oplus (I_{c+d} \otimes A^t)]S^{-1}V,$$

where $a, b, c, d \in \mathbb{N}$ satisfy $(a+b+c+d)n = m$.

Theorem 8 Let $H_1 \in M_n$ and $H_2 \in M_m$ be invertible Hermitian matrices such that H_1 has inertia (p, q) and H_2 has inertia (r, s). The following conditions are equivalent.

(a) There exists a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathcal{U}(H_1)) \subseteq \mathcal{U}(H_2)$.

(b) There exists an invertible matrix $S \in M_m$ satisfying

$$S^* H_2 S = [(I_a \oplus -I_b) \otimes H_1] \oplus [(I_c \oplus -I_d) \otimes (H_1^{-1})^t].$$

(c) There are nonnegative integers u and v such that $(r, s) = u(p, q) + v(q, p)$.

(d) Either (i) $p - q = r - s = 0$ and $(u + v)p = r$, or (ii) $p \neq q$ and $(u, v) = (pr - qs, ps - qr)/(p^2 - q^2)$ is a pair of nonnegative integers.

Proof.

1. Reduction to the case when $(H_1, H_2) = (J_{p,q}, J_{r,s})$.
2. Replacing ϕ by $A \mapsto \phi(I_n)^{-1}\phi(A)$ so that $\phi(I_n) = I_m$.
3. For diagonal matrices D, we have $\phi(D) = U^*(I_t \otimes D)U$ for some unitary $U \in M_m$.
4. Show that for symmetric $A \in M_n$,

$$\phi(A) = U^*[(I_p \otimes A) \oplus (I_q \otimes A^t)]U$$

   for some unitary $U \in M_m$.
5. Fix up the skew-symmetric matrices.
6. Deduce the relation between H_1 and H_2. ■

Further research

* Study applications of the results, or develop results needed in applications.
* Extend the techniques to study other types of matrices.
* Find shorter conceptual proof of the preserver result.
* Consider the real case.

  [Wei, 1972] Except for $n = 2, 4, 8$, a linear operator $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ satisfies $\phi(\mathcal{U}(I_n)) \subseteq \mathcal{U}(I_n)$ if and only if there are unitary matrices $U, V \in M_n$ such that ϕ has the form

$$A \mapsto UAV \quad \text{or} \quad A \mapsto UA^tV.$$

Thank you for your attention!