The definite generalized eigenvalue problem:

A reworked perturbation theory

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Notation and definitions

Let $A$ and $B$ be $n \times n$ Hermitian matrices. They form a definite pair if the Crawford number

$$c(A, B) = \min\{|x^*(A + iB)x| : x \in \mathbb{C}^n, x^*x = 1\} > 0.$$ 

If a real ordered pair $(a, b)$ satisfies $a^2 + b^2 = 1$ and

$$bAx = aBx$$

for some nonzero $x \in \mathbb{C}^n$, then $(a, b)$ is a normalized generalized eigenvalue of $(A, B)$. 
Remark
If $B = I$, we have the usual eigenvalue problem.
If $B$ is positive definite, we have $bB^{-1}Ax = ax$, which is essentially the usual eigenvalue problem.

Proposition For a definite pair $(A, B)$ there exists an invertible matrix $X$ such that

$$X^*(A + iB)X = \text{diag}(a_1 + ib_1, \ldots, a_n + ib_n),$$

where $(a_j, b_j)$ are normalized generalized eigenvalues of $(A, B)$. 
Some nice properties [Li and Mathias, 1998]

Theorem [Uniqueness]
The normalized eigenvalues are unique up to ordering.

Theorem [max-min characterization]
If $C = A + iB$ and $X^* CX = \text{diag} (e^{i\theta_1}, \ldots, e^{i\theta_n})$ with
$\pi > \theta_1 \geq \cdots \geq \theta_n > 0$ then for any $1 \leq j_1 < \cdots < j_k \leq n$,

$$\theta_{j_1} + \cdots + \theta_{j_k} = \sup_{V_1 \subseteq \cdots \subseteq V_k} \inf_{y_t \in W_t} \sum_{t=1}^{k} \arg(y_t^* C y_t).$$

$sup \quad \inf \quad \sum_{t=1}^{k} \arg(y_t^* C y_t)$. 

$\text{dim } V_t = j_t \quad \text{det}(y_r^* y_s) > 0$
Theorem [Interlacing inequalities]
Suppose $C = A + iB$ and $X^*CX = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with $\pi > \theta_1 \geq \cdots \geq \theta_n > 0$. If $Z$ is $n \times n - 1$ such that $Z^*Z$ is invertible, and $Z^*CZ = \text{diag}(e^{i\phi_1}, \ldots, e^{i\phi_{n-1}})$ with $\pi > \phi_1 \geq \cdots \geq \phi_{n-1} > 0$, then

$$\theta_1 \geq \phi_1 \geq \theta_2 \geq \cdots \theta_{n-1} \geq \phi_{n-1} \geq \theta_n.$$
Perturbation bounds

Let \( r(E + iF) = \max\{|x^*(E + iF)x| : x \in \mathbb{C}^n, x^*x = 1\} \) be the numerical radius of \( E + iF \).

**Proposition** Let \((A, B)\) be a definite pair, and \((E, F)\) be a Hermitian pair so that

\[
\varepsilon = \frac{r(E + iF)}{c(A, B)} < 1.
\]

Then \((\tilde{A}, \tilde{B}) = (A, B) + (E, F)\) is a definite pair.

**Theorem** [Li and Mathias, 2004] Let \((A, B)\) be a definite pair. Then

\[
c(A, B) = \inf\{r(E + iF) : (A + E, B + F) \text{ is indefinite}\}.
\]
Suppose $s_1(X) \geq \cdots \geq s_n(X)$ are the singular values of $X$. Denote by $s(X) = (s_1(X), \ldots, s_n(X))$.

For $x, y \in \mathbb{R}^{1 \times n}$, $x \prec_w y$ if the sum of the $k$ largest entries of $x$ is not larger than that of $y$ for $k = 1, \ldots, n$.

**Theorem** Let $(A, B)$ be a definite pair, and $(E, F)$ be a Hermitian pair so that $\varepsilon = r(E + iF)/c(A, B) < 1$.

The matrices $A + iB$ and $\tilde{A} + i\tilde{B}$ are $*$-congruent to

$$e^{\phi} \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \quad \text{and} \quad e^{\phi} \text{diag}(e^{i\tilde{\theta}_1}, \ldots, e^{i\tilde{\theta}_n})$$

such that $\pi \geq \theta_1 \geq \cdots \geq \theta_n > 0$, $\pi > \tilde{\theta}_1 \geq \cdots \geq \tilde{\theta}_n > 0$,

$$(|\tilde{\theta}_1 - \theta_1|, \ldots, |\tilde{\theta}_n - \theta_n|) \prec_w \left(1 + \frac{1}{2\sqrt{1-\varepsilon}}\right) \frac{1}{c(A,B)} s([E|F])$$.
A symmetric norm (symmetric gauge function) on $\mathbb{R}^{1 \times n}$ is an absolute norm satisfying $\|x\| = \|xP\|$ for any permutation matrix $P$. Examples include $\ell_p$ norms with $p \in [1, \infty]$.

Let $x, y \in \mathbb{R}^{1 \times n}$. Then $x \prec_w y$ if and only if $\|x\| \leq \|y\|$ for all symmetric norms $\| \cdot \|$.

**Corollary** Continue to use the hypotheses and notation in the last theorem. For any symmetric norm $\| \cdot \|$ on $\mathbb{R}^n$,

$$\|(\tilde{\theta}_1 - \theta_1), \ldots, (\tilde{\theta}_n - \theta_n)\| \leq \left(1 + \frac{1}{2\sqrt{1-\varepsilon}}\right) \frac{1}{c(A,B)} s([E|F]) \|.$$
A new approach

Proposition Suppose \((A, B)\) is a definite pair. There is an invertible \(X\) with unit columns such that

\[
X^*(A + iB)X = \text{diag}(a_1 + ib_1, \ldots, a_n + ib_n). \tag{1}
\]

Remarks

(1) Note that \((a_j, b_j)\) are generalized eigenvalue pairs, not necessarily normalized.

(2) The pairs \((a_j, b_j)\) are not determined uniquely. They will be if additional assumption is imposed on \(X\).

For example, we may use any one of the following:

(i) \(X\) is chosen so that \((a_j, b_j) = (a_k, b_k)\) whenever

\[
\arg(a_j + ib_j) = \arg(a_k + ib_k).
\]

(ii) \(\det(X^*X)\) is max (min) among all \(X\) satisfying (1).
A new approach

Consider \((\tilde{A}, \tilde{B}) = (A, B) + (E, F)\).

Suppose \(X\) has linearly independent unit columns and \(X^*(A + iB)X = \text{diag}(z_1, \ldots, z_n)\) is diagonal, where \(z_j = d_j e^{i\theta_j}\) for \(j = 1, \ldots, n\), such that \(\pi > \theta_1 > \cdots > \theta_n > 0\).

We continue to assume \(r = r(E + iF) < c(A, B)\).

Applying the mapping \(T \mapsto X^*TX\) to each matrix, we may and we will assume that \((A, B)\) are diagonal matrices.

Note that \(\|E\|\) and \(\|F\|\) will be increased by a factor of \(n\). It still worths the price.
Theorem [Perturbation of normalized generalized eigenvalues]
Set \( u_j = \theta_j + \sin^{-1}(r/d_j) \) and \( l_j = \theta_j - \sin^{-1}(r/d_j) \).

Rearrange the entries of \((u_1, \ldots, u_n)\) and \((l_1, \ldots, l_n)\)
in descending order to get \((\tilde{u}_1, \ldots, \tilde{u}_n)\) and \((\tilde{l}_1, \ldots, \tilde{l}_n)\).
Assume that \((\tilde{A}, \tilde{B})\) has normalized generalized eigenvalues
such that \(\pi > \tilde{\theta}_1 \geq \cdots \geq \tilde{\theta}_n > 0\). Then

\[
\tilde{l}_j \leq \theta_j \leq \tilde{u}_j \quad \text{for } j = 1, \ldots, n.
\]

Consequently,

\[
|\theta_j - \tilde{\theta}_j| \leq \sin^{-1}(r/d_{\text{min}}) \quad \text{for } j = 1, \ldots, n.
\]
Other authors use $c(A, B)$ instead of $d_{\text{min}}$ which may give much worse bounds.

For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$,

then $d_{\text{min}} = \sqrt{1 + \varepsilon^2}$ can be much larger than $c(A, B) = \varepsilon$.

So, $\sin^{-1}(r/d_{\text{min}})$ could be much smaller than $\sin^{-1}(r/c(A, B))$
**Theorem** [A quadratic bound]

Suppose \( E = \begin{pmatrix} 0_m & R \\ R^* & 0_{n-m} \end{pmatrix} \) and \( F = \begin{pmatrix} 0_m & S \\ S^* & 0_{n-m} \end{pmatrix} \).

For any \( k \in \{1, \ldots, n\} \), let

\[
d = \min \{ d_j : j = m + 1, \ldots, n \} \quad \text{and} \quad \Delta_k = \min \{ |d_j \sin(\theta_j - \tilde{\theta}_k)| : j = 1, \ldots, m \}.
\]

Assume that \( \Delta_k > 0 \) and \( r^2/\Delta_k < c(A, B) \). Then

\[
|\theta_k - \tilde{\theta}_k| \leq \sin^{-1} \left( \frac{r^2}{d\Delta_k} \right).
\]
**Eigenvector perturbation**

**Theorem** Suppose \( v \) is a generalized eigenvector corresponding to a normalized generalized eigenvalue \( (\cos \tilde{\theta}, \sin \tilde{\theta}) \) of \( (\tilde{A}, \tilde{B}) \). Let

\[
\tilde{\Delta} = \min \{|d_j \sin(\tilde{\theta} - \theta_k)| : j \neq k\}.
\]

If \( r < \tilde{\Delta} \), then there is \( j \) such that

\[
\|v - e_j\| \leq r / [\tilde{\Delta} + r].
\]
We have the result for the non-diagonal case.

**Theorem** Suppose $A + iB$ is not in diagonal form, and $X$ has linearly independent unit columns $X^*(A + iB)X$ is in diagonal form. Suppose $v$ is a generalized eigenvector corresponding to a normalized generalized eigenvalue $(\cos \tilde{\theta}, \sin \tilde{\theta})$ of $(\tilde{A}, \tilde{B})$. Let

$$\tilde{\Delta} = \min \{|d_j \sin(\tilde{\theta} - \theta_k)| : j \neq k\}.$$

If $\|X\|^2r < \tilde{\Delta}$, then there is $j$ such that

$$\|v - X_j\| \leq \|X\|^2r/(\tilde{\Delta} + \|X\|^2r).$$
There are perturbation bounds on eigensubspace. We omit their discussion because of its technicality.

**Key message**

Our approach can give much stronger bounds.

**Question**

Can one refine the results and get prettier theorems?
Thank you for your attention!