Recent study in preserver problems

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1. Preserver Problems on Matrices/Operators

Let $\mathcal{M}$ be a matrix space or matrix algebra. Characterize functions $\phi : \mathcal{M} \to \mathcal{M}$ with some special properties such as

(a) $f(\phi(A)) = f(A)$ for all $A \in \mathcal{M}$, where f is a given function on $\mathcal{M}$;

(b) $\phi(\mathcal{S}) \subseteq \mathcal{S}$ or $\phi(\mathcal{S}) = \mathcal{S}$ for a certain subset $\mathcal{S} \subseteq \mathcal{M}$;

(c) $\phi(A) \sim \phi(B)$ in $\mathcal{M}$ whenever $A \sim B$ in $\mathcal{M}$ for a certain relations $\sim$ on $\mathcal{M}$.

Very often, ϕ is assumed to be linear, additive, multiplicative, analytic, injective, surjective, unital ....

Why study?

* Interesting results, exceptional maps, proof techniques.
* Better understanding of the analytic, algebraic, geometric properties of the concepts and their interplay.
* Very educational to students and myself!

Examples

Let M_n be the algebra of $n \times n$ complex matrices.

* [Frobenius, 1897] A linear operator $\phi : M_n \to M_n$ satisfies $\det(A) = \det(\phi(A))$ for all $A \in M_n$ if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that ϕ has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN. \tag{S}$$

Let $\text{Eig}\,(A)$ be the set of of eigenvalues of $A \in M_n$.

* [Marcus and Purves, 1959] A linear map $\phi : M_n \to M_n$ satisfies $\text{Eig}\,(\phi(A)) = \text{Eig}\,(A)$ for all $A \in M_n$ if and only if there is an invertible matrix $S \in M_n$ such that ϕ has the form

$$A \mapsto S^{-1}AS \quad \text{or} \quad A \mapsto S^{-1}A^tS.$$

* [Dieudonné, 1949] An invertible linear map $\phi : M_n \to M_n$ mapping the set of singular matrices into itself has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^t N$$

for some $M, N \in M_n$ with $\det(MN) \neq 0$.

* [Marcus and Purves, 1959] A linear map $\phi : M_n \to M_n$ mapping the set of invertible matrices into itself has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^t N$$

for some $M, N \in M_n$ with $\det(MN) \neq 0$.

* [Hiai, 1987] A linear map $\phi : M_n \to M_n$ satisfies the condition that

$$\phi(A) \text{ is similar to } \phi(B) \text{ whenever } A \text{ is similar to } B$$

if and only if there is a fixed $B \in M_n$ such that ϕ has the form

$$A \mapsto (\operatorname{tr} A)B,$$

or there are $a, b \in \mathbb{C}$ and an invertible S such that ϕ has the form

$$A \mapsto aS^{-1}AS + b(\operatorname{tr} A)I$$

or

$$A \mapsto aS^{-1}A^tS + b(\operatorname{tr} A)I.$$

* [Hua, 1951] A bijective map $\phi : M_n \to M_n$ satisfies the condition that

  rank $(\phi(A) - \phi(B)) = 1$ if and only if rank $(A - B) = 1$

  if and only if there are $M, N, R \in M_n$ with $\det(MN) \neq 0$ such that ϕ has the form

$$A \mapsto MAN + R \quad \text{or} \quad A \mapsto MA^tN + R.$$

There are many other results and interesting (difficult) open problems on rank preservers, numerical range preservers, norm preservers, **inertia** preservers, etc.

Let $\mathcal{H}_n$ be the real linear space of $n \times n$ Hermitian matrices. Denote by $\mathcal{H}_n(p,q)$ the inertia class of all matrices in $\mathcal{H}_n$ with p positive and q negative eigenvalues.

Problem Characterize real linear maps ϕ such that

$$\phi(\mathcal{H}_n(p,q)) \subseteq \mathcal{H}_n(p,q).$$

Theorem If (i) $p+q=1$, (ii) $pq \neq 0$ and $p \neq q$, or (iii) $p=q$ is with $n \geq 5p$ or $n = 2p > 2$, then there is an invertible $S \in M_n$ and $\varepsilon \in \{1,-1\}$ such that

$$A \mapsto \varepsilon S^* A S \quad \text{or} \quad A \mapsto \varepsilon S^* A^t S.$$

Remarks

* If we assume that ϕ is invertible or $\phi(\mathcal{H}_n(p,q)) = \mathcal{H}_n(p,q)$, then the result holds for any nonzero pair (p,q).

* If $p = n$ or $q = n$ these reduce to the study of positive linear maps, which have very complicated structure. Similar difficulty if $p = 0$ or $q = 0$ unless $p + q = 1$.

2. A new direction

Problem Study linear preservers $\phi : \mathcal{M} \to \mathcal{M}'$.

Let $\mathcal{U}_n$ be the group of unitary matrices in M_n.

Theorem [Cheung and Li, 2003]
There is a linear map $\phi : M_n \to M_m$ such that $\phi(\mathcal{U}_n) \subseteq \mathcal{U}_m$ if and only if m is a multiple of n and there are $U, V \in \mathcal{U}_m$ such that ϕ has the form

$$A \mapsto U[(I_r \otimes A) \oplus (I_s \otimes A^t)]V.$$

Remarks

* When $n = m$, the result reduces to that in [Marcus, 1959].
* The result has been extended to J-unitary matrices in [Li and Sze, 2004].
* In the real case, when $n = m$ one has to assume $\phi(O_n) = O_n$ when $n = 2, 4, 8$. Otherwise, there are singular linear maps, which are also characterized in [Wei, 1978].
* The problem of characterizing $\phi(GL_n) \subseteq GL_m$ is very difficult.

Problem Prove the real analog of the result of Cheung & Li.

Theorem [Li, Rodman, and Semrl, 2002]

A linear map $\phi : M_{m,n} \to M_{p,q}$ sending rank one matrices to rank one matrices if and only if

(a) there are full rank matrices P and Q of appropriate sizes such that ϕ has the form

$$A \mapsto PAQ \quad \text{or} \quad A \mapsto PA^tQ.$$

(b) $m+n-1 \leq q$ and there is a linear map $f : M_{m,n} \to \mathbb{C}^{1\times q}$ mapping rank one matrices to nonzero vectors such that ϕ has the form

$$A \mapsto Pf(A).$$

(c) $m+n-1 \leq p$ and there is a linear map $g : M_{m,n} \to \mathbb{C}^{p\times 1}$ mapping rank one matrices to nonzero vectors such that ϕ has the form

$$A \mapsto g(A)Q.$$

Remarks

* The result is true for other fields including all algebraically closed fields.
* When $(m, n) = (p, q)$, the result reduces to that in [Marcus and Moyls, 1959].
* There are results and many questions concerning mappings ϕ such that

  $\operatorname{rank} A = k \Rightarrow \operatorname{rank} \phi(A) = s$,

  $\operatorname{rank} A = k \iff \operatorname{rank} \phi(A) = s$,

  $\operatorname{rank} A \leq k \Rightarrow \operatorname{rank} \phi(A) \leq s$,

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  $\operatorname{rank} A = k \Rightarrow \operatorname{rank} \phi(A) \leq s$.

Theorem [Cheung, Li, and Poon, 2003]
Suppose $m \leq 2n-2$. If $\phi : M_n \to M_m$ satisfies $\|\phi(A)\| = \|A\|$ for all $A \in M_n$, then there is $U, V \in M_m$ and a contractive linear map $f : M_n \to M_{m-n}$ such that ϕ has the form

$$A \mapsto U[A \oplus f(A)]V \quad \text{or} \quad A \mapsto U[A^t \oplus f(A)]V.$$

Remarks

* If $m \geq 2n - 1$ there are mappings of the form

$$A \mapsto P^*[(I_r \otimes A) \oplus (I_s \otimes A^t)]Q$$

for some P, Q such that $P^*P = Q^*Q = I_m$.

Question: Are there any other maps?

* There are results on Ky Fan k-norms by [Li, Poon, and Sze, 2004].

  The structure is much tractable if $k > 1$.

Question: How about other unitarily invariant norms?

* There are results on the numerical range, numerical radius, k-numerical radius, etc. In all these results, there are restrictions on dimensions.

Question: How to remove such restrictions?

3. Other new directions

Problem Study additive/multiplicative/continuous preservers

Examples

* [CLS, 2004] Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}$. Then a linear map $\phi : M_{m,n} \to M_{m,n}$ satisfies

$$\|\phi(A) - \phi(B)\| = \|A - B\| \quad \text{for all } A, B \in M_{m,n}$$

if and only if there are $U \in \mathcal{U}_m$, $V \in \mathcal{U}_n$, and $S \in M_{m,n}$ such that

(a) ϕ has the form

$$A \mapsto UAV + S \quad \text{or} \quad A \mapsto U\overline{A}V + S.$$

(b) $m = n$ and ϕ has the form

$$A \mapsto UA^tV + S \quad \text{or} \quad A \mapsto UA^*V + S.$$

Remark Suppose F is a given function on $\mathcal{M}$.

* There are study of $F(A-B) = F(\phi(A) - \phi(B))$ for all $A, B \in \mathcal{M}$.
* There are study of $F(A+tB) = F(\phi(A) + t\phi(B))$ for all $A, B \in \mathcal{M}$ and $t \in T$, where T is a certain subset of $\mathbb{F}$.
* There are study of $F(AB) = F(\phi(A)\phi(B))$ for all $A, B \in \mathcal{M}$.
* One may consider various spaces $\mathcal{M}$ such as rectangular matrices, triangular (block) matrices, symmetric matrices, Hermitian matrices, $B(H)$, $B(X)$, nest algebras, etc.

4. Some additional comments

Studying preserver problems is like digging gold mines.
[A comment of Danny Hershkowitz.]

One may find many interesting problems, results, and techniques. It is easily get addicted!

To advance the subject, it would be desirable to develop new ideas and use techniques from other areas such as group theory, geometry, etc. to solve specific (difficult) problems, or general classes or problems.
It is also desirable to prove some basice preserver results so that other problems can be reduced to them.

THANK YOU FOR YOUR ATTENTION!