Possible Undergraduate Research Projects **HKIEd 2004**

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Preserver Problems

Study mappings from a matrix space to a matrix space with some special properties.

Exercise: Let $S \in M_n$ be invertible. Define $L : M_n \to M_n$ by

$$L(A) = S^{-1}AS \qquad \text{for all } A \in M_n.$$

Prove that

(a) L is linear, i.e., $L(A + B) = L(A) + L(B)$ and $L(tA) = tL(A)$.

(b) L is multiplicative, i.e., $L(AB) = L(A)L(B)$.

(c) A and $L(A)$ always have the same eigenvalues, determinant, rank, ...

Questions:

1. Characterize additive/linear/multiplicative maps $L : M_n \to M_n$ such that A and $L(A)$ always have the same eigenvalues (determinant, rank, ...).
2. What about additive/linear/multiplicative maps on triangular matrices, nonnegative matrices, positive matrices, rectangular matrices? [Try multiplicative preservers of rank/determinant/eigenvalues on 2×2 matrices!]

Motivations for studying these problems.

Results are elegant, problems are natural, proof techniques are inspiring, the study may lead to other interesting subjects.

Some successful projects

* Isometries.

* Linear preservers of generalized numerical ranges and radii.

* Non-negative matrix set preservers (linear or multiplicative).

Matrix inequalities and optimization problems

1. (Numerical linear algebra) Suppose $A \in M_4(\mathbf{R})$ such that $A = A^t$, $A^2 = A$, and $\operatorname{tr} A = 2$. Show that there is a 2×2 principal submatrix of A with all eigenvalues larger than or equal to $1/4$.
2. (Quantum Dynamics) Prove that

$$\max\{\,|\operatorname{tr} C^*UAU^*| : U \text{ unitary}\} = 3\sqrt{3}/2$$

if

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

3. (Positive semi-definite programming) Suppose S is $n \times n$ symmetric with smallest eigenvalue equal to 1. Characterize diagonal matrices with positive diagonal entries D such that $D^{-1}SD + DSD^{-1}$ has minimum eigenvalue ≥ -2 (or any $r \in .$).
4. (Operator theory) Suppose U is unitary. Prove that the largest eigenvalue of $D(U + U^*)D$ is not larger than that of $D^2U + U^*D^2$.
5. (Polynomials) Suppose $u_0, u_1, \ldots, u_n \in \mathbf{D}$, where

$$\mathbf{D} = \{z \in \mathbf{C} : |z| \leq 1\}.$$

Let $U = \operatorname{diag}(u_1, \ldots, u_n)$ and let J_n be the $n \times n$ matrix with all entries equal to one. Show that $(U - u_0 I_n)(I_n - J_n/(n+1))$ has an eigenvalue in $\mathbf{D}$.

<u>Some successful projects</u>

* max/min norms and condition numbers for matrices with prescribed determinant and Frobenius norm.
* max/min determinant of certain $(0, 1)$ matrices with fixed row sums and column sums.

Subspaces of matrices with special (rank) properties

There has been a lot of interest in constructing subspaces of $m \times n$ ($n \times n$ real symmetric, Hermitian) matrices with certain rank (other general) properties such as every nonzero matrix in the subspace has rank at most / at least / exactly k for a fixed $k \leq \min\{m, n\}$.

Special cases include: singular matrix space, nonsingular matrix space, rank k space.

One also wants to find the maximum dimension of such a subspace.

Some specific problems

1. Show that any nonzero real linear combination of the following matrices has rank at least 3:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

2. Show that any nonzero real linear combination of the following matrices has rank at least 4:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 & -1 \end{pmatrix}.$$

Research papers / REU reports / Honors Theses

Names of undergraduate students are in italics.

[AM] *M. Alwill*, C.K. Li, *C. Maher*, and N.S. Sze, Isomorphisms of semigroups of nonnegative matrices, in preparation (based on an REU report).

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[B] *E. Bellenot*, Effects of Biological Invasions on Ecological Communities, REU report (Advisors: C.K. Li and S. Schreiber), William and Mary, 2003.

[Bet] V. Bolotnikov, C.K. Li, *P. Meade*, C. Mehl, and L. Rodman, Shells of matrices in indefinite inner product spaces, Electronic Linear Algebra 9 (2002), 67-92.

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[KL] *A.-L. S. Klaus* and C.K. Li, Isometries for the vector (p, q) norm and the induced (p, q) norm, Linear and Multilinear Algebra 38 (1995), 315-332,(based on an Honors thesis).

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[LMR2] C.K. Li, *P. Mehta*, and L. Rodman, A generalized numerical range: The range of a constrained sesquilinear form, Linear and Multilinear Algebra 37 (1994), 25-50, (based on an REU project).

[LNa] C.K. Li and *S. Nataraj*, Some matrix techniques in game theory, Mathematical Inequalities and Applications 3 (2000),133-141,(based on a Wilson interdisciplinary research project).

[LN] C.K. Li and *I. Nelson*, Perfect Codes on the Towers of Hanoi Graph, Bulletin of the Australian Math. Soc. 57 (1998), no. 3, 367-376, (based on an Honors thesis).

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