Numerical Ranges, Numerical Radii, and Multiplicative Preservers

Chi-Kwong Li
Department of Mathematics
College of William and Mary
Williamsburg, VA 23185

Based on some joint work with

Wai-Shun Cheung (HKU), Antonia Duffner (U. of Lisbon),

Shaun Fallat (U. of Regina), Robert Guralnick (USC),

Leiba Rodman (W&M).

1. Preserver Problems on Matrices/Operators

Let $\mathcal{M}$ be a matrix space or matrix algebra. Characterize functions $\phi : \mathcal{M} \to \mathcal{M}$ with some special properties such as

(a) $f(\phi(A)) = f(A)$ for all $A \in \mathcal{M}$, where f is a given function on $\mathcal{M}$;

(b) $\phi(\mathcal{S}) \subseteq \mathcal{S}$ or $\phi(\mathcal{S}) = \mathcal{S}$ for a certain subset $\mathcal{S} \subseteq \mathcal{M}$;

(c) $\phi(A) \sim \phi(B)$ in $\mathcal{M}'$ whenever $A \sim B$ in $\mathcal{M}$ for a certain relations $\sim$ on $\mathcal{M}$.

Very often, ϕ is assumed to be linear, additive, *multiplicative*, analytic, injective, surjective, unital ....

Why study?

* Interesting results, exceptional maps, proof techniques.
* Better understanding of the analytic, algebraic, geometric properties of the concepts and their interplay.
* Very educational to students and myself!

Examples

Let M_n be the algebra of $n \times n$ complex matrices.

* [Frobenius, 1879] A linear operator $\phi : M_n \to M_n$ satisfies $\det(A) = \det(\phi(A))$ for all $A \in M_n$ if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that ϕ has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN. \tag{S}$$

* [Dieudonné, 1949] An invertible linear map $\phi : M_n \to M_n$ mapping the set of singular matrices into itself has the form (S) for some $M, N \in M_n$ with $\det(MN) \neq 0$.

* [Hua, 1951] A transformation $\phi : M_n \to M_n$ satisfies $\operatorname{rank}(\phi(A) - \phi(B)) = 1$ whenever $\operatorname{rank}(A - B) = 1$ if and only if there are $M, N, R \in M_n$ with $\det(MN) \neq 0$ such that ϕ has the form

$$A \mapsto MAN + R \quad \text{or} \quad A \mapsto MA^tN + R.$$

2. Multiplicative Preservers on SL_n, GL_n, and $M_n^{(k)}$

Let $\mathcal{S} \subseteq M_n$ be a semigroup/group. For examples,
SL_n: group of matrices in M_n with determinant equal to 1.
GL_n: group of invertible matrices.
$M_n^{(k)}$: semigroup of matrices in M_n with rank at most k.

By the results in [Jodeit & Lam, 1969] and [Borel & Tits, 1978], under certain assumptions, a multiplicative map $\phi : \mathcal{S} \rightarrow M_n$ has the form

$$A = (a_{ij}) \mapsto f(\det(A))S^{-1}(\sigma(a_{ij}))S \tag{M1}$$

or

$$A = (a_{ij}) \mapsto f(\det(A))S^{-1}\tau(\sigma(a_{ij}))S, \tag{M2}$$

where $f : \mathbb{C} \rightarrow \mathbb{C}$ is a multiplicative map, $S \in SL_n$, $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ is a field isomorphism, and τ is the mapping $X \mapsto \operatorname{adj}(X)^t$.

3. Multiplicative preservers of numerical range/radius

The *numerical range* and *numerical radius* of $A \in M_n$ are

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\}$$

and
$$r(A) = \{|\mu| : \mu \in W(A)\}.$$

Theorem *Let $\phi : \mathcal{S} \to M_n$ be a multiplicative map. Then $W(\phi(A)) = W(A)$ for all $A \in \mathcal{S}$ if and only if ϕ has the form*

$$A \mapsto U^*AU \quad \text{for some unitary } U;$$

$r(\phi(A)) = r(A)$ for all $A \in \mathcal{S}$ if and only if ϕ has the form

$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*\overline{A}U \quad \text{for some unitary } U.$$

Key steps:

1. Consider special $\phi(D)$ for some D to conclude that (M2) does not hold.
2. S is unitary, g is the identity map, σ has the form $z \to z$ or $z \to \bar{z}$. ■

4. C-Numerical Ranges and Radii

Let $C \in M_n$. The *C-numerical range* and *C-numerical radius* of $A \in M_n$ are

$$W_C(A) = \{\mathrm{tr}\,(CU^*AU) : U \text{ is unitary }\},$$

and
$$r_C(A) = \max\{|\mu| : \mu \in W_C(A)\}.$$

Proposition *Let $A \in M_n$. A compact subset $\mathcal{R}$ of $\mathbb{C}$ equals $W_C(A)$ for some $C \in M_n$ if and only if $\mathcal{R}$ is the image of $\mathcal{U}(A) = \{UAU^* : U$ is unitary$\}$ under a linear functional on M_n.*

Proposition *Let $C \in M_n$. Then r_C is a unitary similarity invariant (usi) semi-norm. It is a norm if and only if* $\mathrm{tr}\, C \neq 0$ *and C is non-scalar.*

Proposition *For every usi norm $\|\cdot\|$ on M_n, there is a compact subset $\mathcal{R} \subseteq M_n$ such that*

$$\|A\| = \max\{r_C(A) : C \in \mathcal{R}\}.$$

In physics, to study the efficiency of polarization or coherence transfer between quantized states under unitary transformations, one needs to study the C-numerical radius of certain nilpotent matrices C and A.

Reference

S.G. Glaster, T. Schulte-Herbrüggen, M. Sievking, O. Schedletzky, N.C. Nielsen, O.W. Sorensen, C. Griesinger. Unitary control in quantum ensembles: Maximizing signal intensity in coherent spectroscopy, Science 280(1998), 421-424.

Theorem *Let* $\mathcal{S} = SL_n$ *or* GL_n, *and let* $C \in M_n$ *be non-scalar. A multiplicative map* $\phi : \mathcal{S} \to M_n$ *satisfies*

$$W_C(\phi(A)) = W_C(A) \text{ for all } A \in \mathcal{S}$$

if and only if there is a unitary U *and a multiplicative map* $f : \mathbb{C}^* \to \mathbf{T}$ *such that one of the following holds.*

(a) ϕ *has the form* $A \mapsto UAU^*$.

(b) $\mu C \in \mathcal{U}(C)$ *for all* $\mu \in \mathbf{T}$ *and*
ϕ *has the form* $A \mapsto f(\det(A))UAU^*$.

(c) $\mu C \in \mathcal{U}(C)$ *for all* $\mu \in \mathbf{T}$, $\tilde{\mu}\overline{C}$ *for some* $\tilde{\mu} \in \mathbf{T}$, *and*
ϕ *has the form* $A \mapsto f(\det(A))U\overline{A}U^*$.

Theorem *Let* $\mathcal{S} = SL_n$ *or* GL_n*, and let* $C \in M_n$ *be non-scalar. A multiplicative map* $\phi : \mathcal{S} \to M_n$ *satisfies*

$$r_C(\phi(A)) = r_C(A) \text{ for all } A \in \mathcal{S}$$

if and only if there is a unitary U *and a multiplicative map* $f : \mathbb{C}^* \to \mathbf{T}$ *such that one of the following holds:*

(a) ϕ *has the form* $A \mapsto f(\det(A))UAU^*$.

(b) *There exists* $\mu \in \mathbf{T}$ *such that* $\mu\overline{C} \in \mathcal{U}(C)$*, and* ϕ *has the form* $A \mapsto f(\det(A))U\overline{A}U^*$.

A block matrix $(X_{ij}) \in M_n$ is in *block shift form* if all the diagonal blocks are square matrices and $X_{ij} = 0$ whenever $j \neq i+1$. This is a generalization of the *weighted shift* matrix where all X_{ij} are one by one.

Proposition *Let $C \in M_n$ be non-scalar. TFAE.*

(a) *C is unitarily similar to a matrix in block shift form.*

(b) *$W_C(A)$ is a circular disk centered at 0 for all $A \in M_n$.*

(c) *$W_C(C^*)$ is a circular disk centered at 0.*

Questions

1. What is the radius of the circular disk?
2. What if $W_C(A)$ is always a circular disk (not necessarily centered at the origin)?

Remark It is interesting to characterize $C \in M_n$ which is unitarily similar to $\mu\overline{C}$ for some $\mu \in \mathbf{T}$ (as well as unitarily similar to a matrix in block shift form).

Example If C is a real matrix in block shift form, then C is unitarily similar to $\mu\overline{C}$ for all $\mu \in \mathbf{T}$.

Example If $C = E_{n1} + \sum_{j=1}^{n-1} E_{j,j+1}$, then C is not in block shift form and C is unitarily similar to $\mu\overline{C}$ for all nth root of unity $\mu \in \mathbf{T}$.

Remark There is an example such that C is in block shift form, but C is not unitarily similar to $\mu\overline{C}$ for any $\mu \in \mathbf{T}$.

Example Let C_1, C_2, C_3 be 2×2 positive definite matrices such that

$$C_1 C_2 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a and c are distinct positive numbers and $b \neq 0$ is real, and the off-diagonal entries of C_3 are non-real. Then $\mathrm{tr}\,(C_1 C_2 C_3) \notin \mathbb{R}$. Next, let B_1, B_2 and B_3 be such that

$$B_2 B_2^* = C_1, \quad B_1^* B_1 = C_2, \quad B_2 B_3^* B_3 B_2^* = C_3,$$

and finally

$$C = \begin{pmatrix} 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then , C cannot be unitarily similar to $\mu \overline{C}$ for any $\mu \in \mathbf{T}$.

Theorem *Let* $C \in M_n$ *be non-scalar, and let* $F_C(A) = W_C(A)$ *or* $r_C(A)$. *A multiplicative map* $\phi : M_n^{(k)} \rightarrow M_n$ *satisfies*

$$F_C(\phi(A)) = F_C(A) \text{ for all } A \in M_n^{(k)}$$

if and only if there is a unitary $U \in SL_n$ *such that one of the following conditions holds:*

(a) ϕ *has the form* $A \mapsto UAU^*$.

(b) $F_C(A) = F_C(\overline{A})$ *for all* $A \in M_n^{(k)}$, *and* ϕ *has the form* $A \mapsto U\overline{A}U^*$.

Proposition *Let* $\Psi_n^{(k)}$ *be the set of matrices* $C \in M_n$ *such that*

$$W_C(A) = W_C(\overline{A}) \quad \textit{for all} \quad A \in M_n^{(k)}.$$

Then

$$\Psi_n^{(n)} \subseteq \Psi_n^{(n-1)} \subseteq \cdots \subseteq \Psi_n^{(1)}.$$

(a) *Suppose* C *has rank at most* k. *Then* $C \in \Psi_n^{(k)}$ *if and only if* C *is unitarily similar to a block shift matrix as well as unitarily similar to* $\mu\overline{C}$ *for some* $\mu \in \mathbf{T}$. *Consequently,* $\Psi_n^{(n)}$ *consists of those* $C \in M_n$ *such that* C *is unitarily similar to a block shift matrix as well as to* $\mu\overline{C}$ *for some* $\mu \in \mathbf{T}$.

(b) *Assume that* $8k \leq n$, *and suppose* C *is unitarily similar to* $(C_1 \otimes I_{4k}) \oplus C_2$ *with* $C_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ *and* $W(C_2) \subseteq W(C_1)$. *Then* $C \in \Psi_n^{(k)}$.

Proposition *Let $\psi_n^{(k)}$ be the set of matrices $C \in M_n$ and*

$$r_C(A) = r_C(\overline{A}) \quad \text{for all} \quad A \in M_n^{(k)}.$$

Then

$$\psi_n^{(n)} \subseteq \psi_n^{(n-1)} \subseteq \cdots \subseteq \psi_n^{(1)} = M_n.$$

(a) *Suppose C has rank at most k. Then $C \in \psi_n^{(k)}$ if and only if C is unitarily similar to $\mu\overline{C}$ for some $\mu \in \mathbf{T}$. Consequently, $\psi_n^{(n)}$ consists of those $C \in M_n$ such that C and $\mu\overline{C}$ are unitarily similar for some $\mu \in \mathbf{T}$.*

(b) *Assume $8k \leq n$, and suppose C is unitarily similar to $(C_1 \otimes I_{4k}) \oplus C_2$ with $C_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $W(C_2) \subseteq W(C_1)$. Then $C \in \psi_n^{(k)}$.*

5. Decomposable numerical ranges and radii

Let $V = \mathbb{C}^n$.

Let H be a subgroup of the symmetric group of degree m.

Suppose $\chi : H \to \mathbb{C}$ is a character of degree 1 on H.

The *symmetrizer* defined by H and χ on the tensor space $\otimes^m V$:

$$S(v_1 \otimes \cdots \otimes v_m) = \tfrac{1}{|H|} \textstyle\sum_{\sigma \in H} \chi(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.$$

The *symmetry class of tensors* over V associated with H and χ is the linear span of $S(\otimes^m V)$.

Then *decomposable tensors* in $V_\chi^m(H)$ are the vectors of the form $S(v_1 \otimes \cdots \otimes v_m)$, denoted by $v_1 * \cdots * v_m$.

Let $A \in M_n$. There is a (unique) *induced matrix* $K(A)$ acting on $V_\chi^m(H)$ such that

$$K(A)v_1 * \ldots * v_m = Av_1 * \cdots * Av_m.$$

Examples

1. If $H = \{e\}$ in S_m and $\chi = \mathbf{1}$ is the principal character, then $K(A) = \otimes^m A$ acts on $\otimes^m \mathbb{C}^n$.
2. If $H = S_m$, $\chi = \mathbf{1}$, then $K(A)$ is the mth induced power acting on the mth completely symmetric space.
3. If $H = S_m$ and χ is the alternate character, then $K(A)$ is the mth compound matrix acting on the mth exterior space.

The *decomposable numerical range* and *decomposable numerical radius* of A are

$$W_\chi(A) = \{(K(A)x^*, x^*) : \\ x^* \text{ is a decomposable unit tensor } \}$$

and

$$r_\chi(A) = \max\{|\mu| : \mu \in W_\chi(A)\}.$$

Theorem *Let $\mathcal{S} = SL_n$ or GL_n. Suppose χ is not of the determinant type, and $F(A) = W(K(A))$ or $W_\chi(A)$. A multiplicative map $\phi : \mathcal{S} \to M_n$ satisfies*

$$F(\phi(A)) = F(A) \quad \text{for all } A \in \mathcal{S}$$

if and only if there is a unitary $U \in SL_n$ such that ϕ has the

$$A \mapsto U^* AU.$$

Theorem *Let $\mathcal{S} = SL_n$ or GL_n. Suppose χ is not of the determinant type. Let $F(A) = r(K(A))$ or $r_\chi(A)$. A multiplicative map $\phi : \mathcal{S} \to M_n$ satisfies*

$$F(\phi(A)) = F(A) \quad \textit{for all } A \in \mathcal{S}$$

if and only if there exist a multiplicative map $g : \mathbb{C}^* \to \mathbf{T}$ and a unitary $U \in SL_n$ such that one of the following holds.

(a) *ϕ has the form*

$$A \mapsto g(\det(A))U^*AU \quad \textit{or} \quad A \mapsto g(\det(A))U^*\overline{A}U.$$

(b) $F(A) = F(\det(A)^{2/n}\tau(A))$ *and ϕ has the form*

$$A \mapsto g(\det(A))|\det(A)|^{2/n}U^*\tau(A)U$$

or

$$A \mapsto g(\det(A))|\det(A)|^{2/n}U^*\tau(\overline{A})U.$$

Theorem *Suppose* χ *is not of the determinant type,* $\mu(\overline{\Delta}) < n$ *and* $k \geq \mu(\overline{\Delta})$. *Let* $\mathcal{R}$ *be a semigroup in* M_n *containing* $M_n^{(k)}$, *and let* $F(A) = r(K(A))$, $r_\chi(A)$, $W(K(A))$, *or* $W_\chi(A)$ *A multiplicative map* $\phi : \mathcal{R} \to M_n$ *satisfies*

$$F(\phi(A)) = F(A) \qquad \text{for all } A \in \mathcal{R},$$

if and only if one of the following holds.

(a) $F(A) = r(K(A))$ *or* $r_\chi(A)$; *there exists a unitary* U *such that* ϕ *has the form*

$$A \mapsto U^*AU \qquad \text{or} \qquad A \mapsto U^*\overline{A}U.$$

(b) $F(A) = W(K(A))$ *or* $W_\chi(A)$; *there exists a unitary* U *such that* ϕ *has the form*

$$A \mapsto U^*AU.$$

6. (p,q) Numerical ranges and radii

Let $1 \leq p \leq q \leq n$ integers be such that $(p,q) \neq (n,n)$. The (p,q)-*numerical range* and (p,q)-*numerical radius* are

$$W_{p,q}(A) = \{E_p(X^*AX) : X \in M_{n\times q},\ X^*X = I_q\},$$

where $E_p(Y)$ denotes the pth elementary symmetric function of the eigenvalues of Y, and

$$r_{p,q}(A) = \max\{|\mu| : \mu \in W_{p,q}(A)\}.$$

Special cases

* $W_{1,1}(A) = W(A)$ and $r_{1,1}(A) = r(A)$;
* $W_{1,q}(A) = \{\mathrm{tr}\,(X^*AX) : X \text{ is } n \times q,\ X^*X = I_q\}$ and $r_{1,q}(A)$ are the qth numerical range and radius considered by Halmos and Marcus, etc.

Theorem *Let* $\mathcal{S} = SL_n$, GL_n, *or* $M_n^{(k)}$. *Suppose* p, q *are integers,* $1 \leq p \leq q \leq n$, *and* $(p,q) \neq (n,n)$. *A multiplicative map* $\phi : \mathcal{S} \to M_n$ *satisfies*

$$W_{p,q}(\phi(A)) = W_{p,q}(A) \text{ for all } A \in \mathcal{S}$$

if and only if there is an $S \in SL_n$ *such that one of the following conditions holds.*

(a) ϕ *has the form* $A \mapsto SAS^{-1}$, *where* S *is unitary if* $n > q$.

(b) $2p = 2q = n > 2$, *and* ϕ *has the form* $A \mapsto S\tau(A)S^{-1}$, *where* S *is unitary.*

(c) $n = q = 2p > 2$, $\mathcal{S} = SL_n$, *and* ϕ *has the form* $A \mapsto S\tau(A)S^{-1}$.

Theorem *Let* $\mathcal{S} = SL_n$, GL_n, *or* $M_n^{(k)}$. *Suppose* p, q *are integers,* $1 \le p \le q \le n$, *and* $(p,q) \neq (n,n)$. *A multiplicative map* $\phi : \mathcal{S} \to M_n$ *satisfies*

$$r_{p,q}(\phi(A)) = r_{p,q}(A) \text{ for all } A \in \mathcal{S}$$

if and only if there is $S \in SL_n$ *such that one of the following conditions holds.*

(a) *There is a multiplicative map* $f : \mathbb{C}^* \to \mathbf{T}$ *such that* ϕ *has the form*

$$A \mapsto f(\det A) SAS^{-1} \quad \text{or} \quad A \mapsto f(\det A) S\overline{A}S^{-1},$$

where S *is unitary if* $n > q$.

(b) $2p = 2q = n > 2$, *and there is a multiplicative map* $f : \mathbb{C}^* \to \mathbf{T}$ *such that* ϕ *has the form*

$$A \mapsto f(\det A)U\tau(A)U^{-1} \quad \textit{or} \quad A \mapsto f(\det A)U\tau(\overline{A})U^{-1},$$

where U *is unitary.*

(c) $2p = q = n > 2$, *and there is a multiplicative map* $f : \mathbb{C}^* \to \mathbb{C}^*$ *such that* $|f(\mu)|^n = |\mu|^2$ *for all* $\mu \in \mathbb{C}^*$ *and* ϕ *has the form*

$$A \mapsto f(\det A)S\tau(A)S^{-1} \quad \textit{or} \quad A \mapsto f(\det A)S\tau(\overline{A})S^{-1}.$$

7. Further research

1. Results on nonnegative and positive matrices.
   Working with Sze.
2. The (p, q) numerical range can be viewed as the pth derivation of the induced matrix A acting on the qth Grassmann space. One may consider results for other derivations of induced operators.
   Plan to work with Tin-Yau Tam.
3. Just assume that $F(\phi(A)\phi(B)) = F(AB)$ for all $A, B \in \mathcal{S}$.
   Working with the HK2004 preserver group.

Thank you for your attention!