Induced operators on symmetry classes of tensors

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Introduction

$V$: an $n$-dimensional inner product space over $\mathbb{C}$.

$\otimes^m V$: the $m$th tensor space of $V$ spanned by

$$v_1 \otimes \cdots \otimes v_m.$$ 

There is an induced inner product on $\otimes^m V$ such that

$$(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^{m} (u_i, v_i).$$
Example If $V = \mathbb{C}^n$, then

$$x \otimes y = \begin{pmatrix} x_1y \\ \vdots \\ x_ny \end{pmatrix} \equiv \begin{pmatrix} x_1y^t \\ \vdots \\ x_ny^t \end{pmatrix} = xy^t.$$ 

$$x \otimes y \otimes z = \begin{pmatrix} x_1(y \otimes z) \\ \vdots \\ x_n(y \otimes z) \end{pmatrix} \equiv \begin{pmatrix} x_1(yz^t) \\ \vdots \\ x_n(yz^t) \end{pmatrix}.$$
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If $V = M_k$ then

$$A \otimes B = (a_{ij}B) \quad \text{and} \quad A \otimes B \otimes C = (a_{ij}(B \otimes C)),$$ 

etc.
For any linear maps $T_j : V \to V$ for $j = 1, \ldots, m$,  

$$(T_1 \otimes \cdots \otimes T_m)(v_1 \otimes \cdots \otimes v_m) = (T_1 v_1) \otimes \cdots \otimes (T_m v_m).$$

In particular, if $T : V \to V$, then

$$\otimes^m T(v_1 \otimes \cdots \otimes v_m) = (Tv_1) \otimes \cdots \otimes (Tv_m).$$
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**Example** If $V = \mathbb{C}^n$, then

$$(T_1 \otimes T_2)(x \otimes y) = (T_1 x) \otimes (T_2 y) \equiv T_1 x y^t T_2^t$$

and

$$(T \otimes T)(x \otimes y) = (Tx) \otimes (Ty) \equiv T x y^t T^t.$$
Let $H < S_m$, the symmetric group of degree $m$. We can use the irreducible characters $\chi : H \to \mathbb{C}$ to do the following orthogonal decompositions:

(a) $\otimes^m V = \text{direct sum of the subspaces } V^m_\chi(H)$, the symmetry classes of tensors;

(b) $\otimes^m T = \text{direct sum of the induced operators } K_\chi(T)$, where $K_\chi(T) = K(T)$ acts on $V^m_\chi(H)$. 
Why bother to study?

1. Symmetry class of tensors arises naturally in the study of many subjects: differential geometry, representation theory, quantum physics, operator theory, combinatorial theory, ....

2. It is helpful to formulate, study, and solve problems of $V$ in terms of $\otimes^m V$ or $V^m_\chi(H)$. 
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Examples

(a) Let $A$ and $B$ be positive definite. Then the Hadamard product $A \circ B = (a_{ij} b_{ij})$ is positive definite.

The matrix $A \circ B$ is a principal submatrix of $A \otimes B = (a_{ij} B)$, which is positive definite with minimum eigenvalue $\lambda_{\min}(A) \lambda_{\min}(B)$. 
(b) Suppose $\mu$ and $\nu$ are algebraic numbers of degree $p$ and $q$, resp., i.e., there are polynomials $f$ and $g$ of degrees $p$ and $q$ with integer coefficients such that $f(\mu) = 0 = g(\nu)$. Then $\mu \nu$ and $\mu + \nu$ are algebraic number of degree at most $pq$. 
(b) Suppose $\mu$ and $\nu$ are algebraic numbers of degree $p$ and $q$, resp., i.e., there are polynomials $f$ and $g$ of degrees $p$ and $q$ with integer coefficients such that $f(\mu) = 0 = g(\nu)$. Then $\mu \nu$ and $\mu + \nu$ are algebraic number of degree at most $pq$.

Let $A \in M_p$ and $B \in M_q$ be the companion matrices of $f$ and $g$, i.e.,

$$f(z) = \det(zI_p - A) \quad \text{and} \quad g(z) = \det(zI_q - B).$$

Then $\mu \nu$ and $\mu + \nu$ are roots of the characteristic polynomials of

$$\det(zI_{pq} - A \otimes B) \quad \text{and} \quad \det(zI - (A \otimes I_q + I_p \otimes B)).$$
(b) Suppose $\mu$ and $\nu$ are **algebraic numbers** of degree $p$ and $q$, resp., i.e., there are polynomials $f$ and $g$ of degrees $p$ and $q$ with integer coefficients such that $f(\mu) = 0 = g(\nu)$. Then $\mu\nu$ and $\mu + \nu$ are algebraic number of degree at most $pq$.

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(c) Let $A, B, C$ be given. Then $AX - XB = C$ is solvable if and only if $C$ is in the range space of $L(X) = AX - XB$. 
(b) Suppose $\mu$ and $\nu$ are algebraic numbers of degree $p$ and $q$, resp., i.e., there are polynomials $f$ and $g$ of degrees $p$ and $q$ with integer coefficients such that $f(\mu) = 0 = g(\nu)$. Then $\mu \nu$ and $\mu + \nu$ are algebraic number of degree at most $pq$.

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Then $\mu \nu$ and $\mu + \nu$ are roots of the characteristic polynomials of $\det(zI_{pq} - A \otimes B)$ and $\det(zI - (A \otimes I_q + I_p \otimes B))$.

(c) Let $A, B, C$ be given. Then $AX - XB = C$ is solvable if and only if $C$ is in the range space of $L(X) = AX - XB$.

The linear map $L$ can be viewed as $A \otimes I - I \otimes B^t$.

It is invertible if and only if $A$ and $B$ have no common eigenvalues.
(d) Let $A \in M_n$ have singular values $s_1 \geq \cdots \geq s_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then

$$|\lambda_1 \cdots \lambda_m| \leq s_1 \cdots s_m, \quad 1 \leq m < n.$$ 

The $m$th compound matrix $C_m(A) \in M_{n\choose m}$ has entries equal to the $m \times m$ minors of $A$ arranged in lexicographic order acting on the $m$th exterior space.

For example, for $A \in M_3$,

$$C_2(A) = \begin{pmatrix}
M_{12,12} & M_{12,13} & M_{12,23} \\
M_{13,12} & M_{13,13} & M_{13,23} \\
M_{23,12} & M_{23,13} & M_{23,23}
\end{pmatrix}.$$ 

Using the fact that $|\lambda_1| \leq s_1$,

$$\lambda_1(C_m(A)) = |\lambda_1 \cdots \lambda_m| \quad \text{and} \quad s_1(C_m(A)) = s_1 \cdots s_m,$$

one gets the result.
The formal definitions

For each $\sigma \in S_m$, define $P(\sigma)$ on $\otimes^m V$:

$$P(\sigma)(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

on the decomposable tensors $v_1 \otimes \cdots \otimes v_m$.

Let $\chi : H \to \mathbb{C}$ be an irreducible character of $H < S_m$. Then the symmetrizer

$$S_\chi := \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma)$$

is an orthogonal projector (Hermitian idempotent).
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The set $V^m_\chi(H) := S_\chi(\otimes^m V)$ is called the symmetry class of tensors over $V$ spanned by decomposable symmetrized tensors:

$$S_\chi(v_1 \otimes \cdots \otimes v_m) = v_1 \ast \cdots \ast v_m.$$
For any $T : V \to V$, the subspace $V^m_\chi(H)$ is stable under $\otimes^m T$. Thus the induced operator

$$K(T) = K_\chi(T) = \otimes^m T|_{V^m_\chi(H)}$$

is the unique induced operator acting on $V^m_\chi(H)$ satisfying

$$K(T)v_1 \ast \cdots \ast v_m = Tv_1 \ast \cdots \ast Tv_m.$$
Examples  Let $V = \mathbb{C}^n$.

(a) Consider $\otimes^2 V$. Identify $x \otimes y$ with $xy^t$.

For $H = S_2$, only two irreducible characters:

The **principal character** $1$ and the **alternate character** $\varepsilon$. We have

$$\otimes^2 V \equiv M_n = S^2_1(V) \oplus S^2_\varepsilon(V),$$

where $S^2_1(V) = \text{span} \{ (uv^t + vu^t)/2 : u, v \in \mathbb{C}^n \}$ (symmetric matrices), $S^2_\varepsilon(V) = \text{span} \{ (uv^t - vu^t)/2 : u, v \in \mathbb{C}^n \}$ (skew-symmetric matrices).
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For any $T : \mathbb{C}^n \to \mathbb{C}^n$, we have

$$\otimes^2 T(xy^t) = Txy^tT^t \quad \text{for all} \ x, y \in \mathbb{C}^n,$$

and

$$\otimes^2 T \equiv K_1(T) \oplus K_\varepsilon(T) = P_2(T) \oplus C_2(T).$$
(b) Let $H = S_m$ and $\chi \equiv \varepsilon$ be the alternate character.

Then $V_{\chi}^m(H)$ is the $m$th exterior space over $\mathbb{C}^n$ of dimension $\binom{n}{m}$, and $K(T) = C_m(T)$ is the $m$th exterior power of $T$.

(c) Let $H = S_m$ and $\chi \equiv 1$ be the principal character.

Then $V_{\chi}^m(H)$ is the $m$th completely symmetric space over $\mathbb{C}^n$, which has dimension $\binom{m+n-1}{m}$, and $K(T) = P_m(T)$ is the $m$th induced power of $T$.

If $A \in M_n$, then $P_m(A)$ is the $m$th permanental compound of $A$ with entries equal to the permanent of

$$A[i_1, \ldots, i_m; j_1, \ldots, j_m]$$

with $1 \leq i_1 \leq \cdots \leq i_m \leq n$ and $1 \leq j_1 \leq \cdots \leq j_m \leq n$. 
**Operator properties of** $T$ **and** $K(T)$

**Theorem** If $T$ has nice (algebraic or analytic) properties, say, $T$ is normal, unitary, positive (semi-)definite, or Hermitian, then so has $K(T)$.

**Question** When does the converse hold?

**Good cases**

If $\chi \equiv 1$, the principal character, the converses are always valid (up to a multiple).

Suppose $\chi \equiv \varepsilon$ on $S_m$ with $m < n$.

Then $K(T)$ is unitary/scalar if and only if $T$ is unitary/scalar;

$\eta K(T)$ is positive definite if and only if $\xi T$ is positive definite with $\xi^m = 1$. 

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Bad cases

Suppose $H = S_m$, and $T = T_1 \oplus 0$ with $T_1$ acting on an $m$-dimensional subspace such that $\det(T_1) \neq 0$. Then $C_m(T)$ is the rank one normal operator with matrix representation
\[
\text{diag}(\det(T_1), 0, \ldots, 0).
\]

So, $C_m(T)$ is Hermitian (positive semi-definite) if and only if $\det(T_1) \in \mathbb{R}$ ($\det(T_1) > 0$).

If $m = n$, then $C_m(T) = [\det(T)]$ is always normal. It is unitary if and only if $|\det(T)| = 1$; it is Hermitian if and only if .....
Results [Li and Zaharia, 2001], [Li and Tam, 2005]

A character \( \chi \) is of the **determinant type** if \( K(T) \equiv (\det T)^r I \) for some positive integer \( r \).

A character \( \chi \) is of the **special type** if \( K(T_1 \oplus 0) \equiv (\det T_1)^r I \oplus 0 \) for some positive integer \( r \).

**Theorem** Suppose \( \chi \) is not of the determinant type. Then \( K(T) \) is nonzero normal if and only if

(i) \( T \) is normal, or

(ii) \( \chi \) is of the special type, and \( T \) is unitarily similar to \( T_1 \oplus 0 \) with an invertible \( T_1 \).
Theorem Suppose $\chi$ is not of the determinant type. Then $K(T)$ is Hermitian (positive semi-definite, a multiple of Hermitian idempotent) if and only if

(i) $\xi T$ has the corresponding property for some $\xi^m = 1$, or

(ii) $\chi$ is of the special type, and $T$ is unitarily similar to $T_1 \oplus 0$ with an invertible $T_1$ such that .....
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Theorem Suppose $\chi$ is not of the determinant type.

(a) $K(T)$ is unitary if and only $T$ is unitary.

(b) $K(T) = \eta I$ is a scalar if and only if $T = \xi I$ with $\xi^m = \eta$.

(c) $\eta K(T)$ is positive definite if and only if there is $\xi \in \mathbb{C}$ with $\xi^m = \eta$ such that $\xi T$ is positive definite.
A related problem

If \( T = \xi S \) for some complex number \( \xi \) with \( \xi^m = 1 \), then \( K(T) = K(S) \). But the converse may not be true.

**Theorem** Suppose \( \chi \) is not of the determinant type. If \( K(T) = K(S) \) is nonzero, then

(i) \( T = \xi S \) with \( \xi^m = 1 \), or

(ii) \( \chi \) is of the special type, there are unitary \( U \) and \( W \) such that \( UTW = T_1 \oplus 0 \) and \( USW = S_1 \oplus 0 \) so that \( \det(T_1)^r = \det(S_1)^r \) is nonzero.
Numerical ranges and decomposable numerical ranges

The numerical range of $T : V \to V$ is defined by

$$W(T) = \{(Tv, v) : v \in V, (v, v) = 1\}.$$
Numerical ranges and decomposable numerical ranges

The numerical range of $T : V \rightarrow V$ is defined by

$$W(T) = \{(Tv, v) : v \in V, (v, v) = 1\}.$$

It is useful in studying $T$. For example,

(a) $W(T)$ is always compact and convex.

(b) $\text{Sp}(T) \subseteq W(T)$.

(c) Conical points of $W(A)$ are reducing eigenvalues of $T$. 
(d) \( T = \mu I \) if and only if \( W(T) = \{\mu\} \).

(e) \( T = T^* \) if and only if \( W(T) \subseteq \mathbb{R} \).

(f) \( T \) is positive semi-definite if and only if \( W(T) \subseteq [0, \infty) \).

(g) \( T \) is positive definite if and only if \( W(T) \subseteq (0, \infty) \).

(h) \( T \) is unitary if and only if \( W(T) \subseteq \mathcal{D} \) and \( \text{Sp}(T) \subseteq \partial \mathcal{D} \), where \( \mathcal{D} = \{\mu \in \mathbb{C} : |\mu| \leq 1\} \).
The decomposable numerical range of $T : V \rightarrow V$ is defined by

$$W_{\chi}(T) = \{(K(T)v^*, v^*) : v^* = v_1 \cdots v_m \text{ has unit length}\}.$$ 

Since not all unit vectors in $V_{\chi}^m(H)$ are decomposable,

$$W_{\chi}(T) \subseteq W(K(T))$$

and the equality does not hold in general.
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Basic properties

(a) Compactness. Yes.

(b) Convexity. No in general.

(c) $\text{Sp}(K(T)) \subseteq W_{\chi}(T)$. Yes.

(d) In some cases, conical points of $W_{\chi}(T)$ are eigenvalues of $K(T)$.
Question Can we deduce properties of $T$ from $W_\chi(T)$?

[Li and Zaharia, 2001]

Theorem Suppose $\chi$ is a linear irreducible character. Then

(e) $W_\chi(T) = \{\eta\}$ if and only if $T = \xi I$ with $\xi^m = \eta$;

(f) $\eta W_\chi(T) \subseteq (0, \infty)$ if and only if $\xi T$ is positive definite for some $\xi$ with $\xi^m = \eta$;

(g) $W_\chi(T) \subseteq \mathbb{D}$ and $\text{Sp}(K(T)) \subseteq \partial \mathbb{D}$ if and only if $T$ is unitary.

(h) $\eta W_\chi(T) \subseteq \mathbb{R}$ if and only if $\xi T$ is self-adjoint for some $\xi$ with $\xi^m = \pm \eta$ or $\chi$ is of the special type and $T$ is unitarily similar to $T_1 \oplus 0$ ....
The proofs used heavily the induced matrix structure of $K(A)$ for $A \in M_n$ if $\chi$ is a linear irreducible character.
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**Open problem** What happen for non-linear characters $\chi : H \to \mathbb{C}$?

Need some new ideas and techniques.
Inequalities

Consider the spectral norm $\|T\| = s_1(T)$,

the numerical radius $r(T) = \max\{|\mu| : \mu \in W(T)\},$

and the spectral radius $\rho(T) = \max\{|\mu| : \mu \in \text{Sp}(T)\}$.

Then

$$|\det(T)|^{1/n} \leq \rho(T) \leq r(T) \leq \|T\| \leq 2r(T).$$

Equality cases may lead to nice algebraic structure of $T$.

**Theorem** $T$ is a multiple of a unitary operator if and only if

$$|\det(T)|^{1/n} = f(T)$$

for $f(T) = r(T)$ or $\|T\|$. 
Question Are there similar results for induced operators?

We have

\[ |\det(T)|^{m/n} \leq \rho(K(T)) \leq r_\chi(T) \leq r(K(T)) \leq \|K(T)\|. \]

[Li and Zaharia, 2001] used induced matrices to obtain results for \( K_\chi(T) \) when \( \chi \) is a linear irreducible character.

Theorem Suppose \( \chi \) is a linear irreducible character not of the determinant type. An operator \( T \) is a multiple of a unitary operator if and only if \( \rho(K(A)) = f(A) \) for \( f(A) = r_\chi(A), r(K(A)) \) and \( \|K(A)\| \).

Not many results for non-linear characters are available yet.
Preserver problems

Let $F(A)$ be a function on $A \in M_n$. We are interested in studying the structure of maps $\phi$ such that $F(\phi(X)) = F(X)$ for all $X \in M_n$.

(a) linear preservers of $\det(A)$ has the form

$$\phi(X) = MXN \text{ or } \phi(X) = MX^tN, \det(MN) = 1.$$ 

(b) linear preservers of $\text{Sp}(A)$ has the form

$$\phi(X) = SXS^{-1} \text{ or } \phi(X) = SX^tS^{-1}, S \text{ invertible}.$$ 

(c) linear preservers of $W(A)$ has the form

$$\phi(X) = UXU^* \text{ or } \phi(X) = UX^tU^*, U \text{ unitary}.$$
(d) linear preservers of $\|A\|$ has the form

$$\phi(X) = U XV \text{ or } \phi(X) = U X^t V, \text{ } U \text{ and } V \text{ are unitary.}$$

(e) linear preservers of $r(A)$ has the form

$$\phi(X) = \mu UXU^* \text{ or } \phi(X) = \mu UXU^*, \text{ } |\mu| = 1, \text{ } U \text{ unitary.}$$

(f) linear preservers of $\rho(A)$, $|\det(A)|$, etc. has the form

$$\ldots$$
There are also results for multiplicative preservers of
\[ \text{det}(A), \text{Sp}(A), W(A), \|A\|, r(A), \rho(A), |\text{det}(A)|. \]

[Li and Zaharia, 2001] and [Cheung, Duffner, and Li, 2005] have obtained results of linear and multiplicative preservers of

\[ \text{Sp}(K(A)), W_\chi(A), \|K(A)\|, r(K(A)), r_\chi(A), \rho(K(A)) \]

for linear irreducible characters. The preservers are often just \( \xi \) multiple of the standard form with \( \xi^m = 1 \).
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**Open problem** How about general preservers?
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There are many other challenging / interesting problems. Any help would be welcome!
Thank you for your attention!