Induced operators on symmetry classes of tensors

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Introduction

V: an n-dimensional inner product space over $\mathbb{C}$.

$\otimes^m V$: the mth tensor space of V spanned by

$$v_1 \otimes \cdots \otimes v_m.$$

There is an induced inner product on $\otimes^m V$ such that

$$(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^{m} (u_i, v_i).$$

Example If $V = \mathbb{C}^n$, then

$$x \otimes y = \begin{pmatrix} x_1 y \\ \vdots \\ x_n y \end{pmatrix} \equiv \begin{pmatrix} x_1 y^t \\ \vdots \\ x_n y^t \end{pmatrix} = x y^t.$$

$$x \otimes y \otimes z = \begin{pmatrix} x_1 (y \otimes z) \\ \vdots \\ x_n (y \otimes z) \end{pmatrix} \equiv \begin{pmatrix} x_1 (y z^t) \\ \vdots \\ x_n (y z^t) \end{pmatrix}.$$

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If $V = M_k$ then

$$A \otimes B = (a_{ij}B) \qquad \text{and} \qquad A \otimes B \otimes C = (a_{ij}(B \otimes C)), \quad \text{etc.}$$

For any linear maps $T_j : V \to V$ for $j = 1, \ldots, m$,

$$(T_1 \otimes \cdots \otimes T_m)(v_1 \otimes \cdots \otimes v_m) = (T_1 v_1) \otimes \cdots \otimes (T_m v_m).$$

In particular, if $T : V \to V$, then

$$\otimes^m T(v_1 \otimes \cdots \otimes v_m) = (Tv_1) \otimes \cdots \otimes (Tv_m).$$

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Example If $V = \mathbb{C}^n$, then

$$(T_1 \otimes T_2)(x \otimes y) = (T_1 x) \otimes (T_2 y) \equiv T_1 x y^t T_2^t$$

and

$$(T \otimes T)(x \otimes y) = (Tx) \otimes (Ty) \equiv T x y^t T^t.$$

Let $H < S_m$, the symmetric group of degree m. We can use the irreducible characters $\chi : H \to \mathbb{C}$ to do the following orthogonal decompositions:

(a) $\otimes^m V =$ direct sum of the subspaces $V_\chi^m(H)$,

the symmetry classes of tensors;

(b) $\otimes^m T =$ direct sum of the induced operators $K_\chi(T)$,

where $K_\chi(T) = K(T)$ acts on $V_\chi^m(H)$.

Why bother to study?

1. Symmetry class of tensors arises naturally in the study of many subjects: differential geometry, representation theory, quantum physics, operator theory, combinatorial theory, ...

2. It is helpful to formulate, study, and solve problems of V in terms of $\otimes^m V$ or $V_\chi^m(H)$.

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Examples

(a) Let A and B be positive definite. Then the Hadamard product $A \circ B = (a_{ij}b_{ij})$ is positive definite.

The matrix $A \circ B$ is a principal submatrix of $A \otimes B = (a_{ij}B)$, which is positive definite with minimum eigenvalue $\lambda_{\text{min}}(A)\lambda_{\text{min}}(B)$.

(b) Suppose μ and ν are algebraic numbers of degree p and q, resp., i.e., there are polynomials f and g of degrees p and q with integer coefficients such that $f(\mu) = 0 = g(\nu)$. Then $\mu\nu$ and $\mu + \nu$ are algebraic number of degree at most pq.

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Let $A \in M_p$ and $B \in M_q$ be the companion matrices of f and g, i.e., $f(z) = \det(zI_p - A)$ and $g(z) = \det(zI_q - B)$.

Then $\mu\nu$ and $\mu + \nu$ are roots of the characteristic polynomials of $\det(zI_{pq} - A \otimes B)$ and $\det(zI - (A \otimes I_q + I_p \otimes B))$.

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(c) Let A, B, C be given. Then $AX - XB = C$ is solvable if and only if C is in the range space of $L(X) = AX - XB$.

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(c) Let A, B, C be given. Then $AX - XB = C$ is solvable if and only if C is in the range space of $L(X) = AX - XB$.

The linear map L can be viewed as $A \otimes I - I \otimes B^t$.

It is invertible if and only if A and B have no common eigenvalues.

(d) Let $A \in M_n$ have singular values $s_1 \geq \cdots \geq s_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then

$$|\lambda_1 \cdots \lambda_m| \leq s_1 \cdots s_m, \qquad 1 \leq m < n.$$

The mth compound matrix $C_m(A) \in M_{\binom{n}{m}}$ has entries equal to the $m \times m$ minors of A arranged in lexicographic order acting on the mth exterior space.

For example, for $A \in M_3$,

$$C_2(A) = \begin{pmatrix} M_{12,12} & M_{12,13} & M_{12,23} \\ M_{13,12} & M_{13,13} & M_{13,23} \\ M_{23,12} & M_{23,13} & M_{23,23} \end{pmatrix}.$$

Using the fact that $|\lambda_1| \leq s_1$,

$$\lambda_1(C_m(A)) = |\lambda_1 \cdots \lambda_m| \quad \text{and} \quad s_1(C_m(A)) = s_1 \cdots s_m,$$

one gets the result.

The formal definitions

For each $\sigma \in S_m$, define $P(\sigma)$ on $\otimes^m V$:

$$P(\sigma)(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

on the decomposable tensors $v_1 \otimes \cdots \otimes v_m$.

Let $\chi : H \to \mathbb{C}$ be an irreducible character of $H < S_m$. Then the symmetrizer

$$S_\chi := \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma)$$

is an orthogonal projector (Hermitian idempotent).

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The set $V_\chi^m(H) := S_\chi(\otimes^m V)$ is called the symmetry class of tensors over V spanned by decomposable symmetrized tensors:

$$S_\chi(v_1 \otimes \cdots \otimes v_m) = v_1 * \cdots * v_m.$$

For any $T : V \to V$, the subspace $V_\chi^m(H)$ is stable under $\otimes^m T$. Thus the induced operator

$$K(T) = K_\chi(T) = \otimes^m T|_{V_\chi^m(H)}$$

is the unique induced operator acting on $V_\chi^m(H)$ satisfying

$$K(T)v_1 * \cdots * v_m = Tv_1 * \cdots * Tv_m.$$

Examples Let $V = \mathbb{C}^n$.

(a) Consider $\otimes^2 V$. Identify $x \otimes y$ with xy^t.

For $H = S_2$, only two irreducible characters:

The principal character $\mathbf{1}$ and the alternate character ε. We have

$$\otimes^2 V \equiv M_n = S^2_{\mathbf{1}}(V) \oplus S^2_{\varepsilon}(V),$$

where $S^2_{\mathbf{1}}(V) = \text{span}\,\{(uv^t + vu^t)/2 : u, v \in \mathbb{C}^n\}$ (symmetric matrices),

$S^2_{\varepsilon}(V) = \text{span}\,\{(uv^t - uv^t)/2 : u, v \in \mathbb{C}^n\}$ (skew-symmetric matrices).

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For any $T : \mathbb{C}^n \to \mathbb{C}^n$, we have

$$\otimes^2 T(xy^t) = Txy^tT^t \qquad \text{for all } x, y \in \mathbb{C}^n,$$

and

$$\otimes^2 T \equiv K_{\mathbf{1}}(T) \oplus K_{\varepsilon}(T) = P_2(T) \oplus C_2(T).$$

(b) Let $H = S_m$ and $\chi \equiv \varepsilon$ be the alternate character.

Then $V_\chi^m(H)$ is the mth exterior space over $\mathbb{C}^n$ of dimension $\binom{n}{m}$

and $K(T) = C_m(T)$ is the mth exterior power of T.

(c) Let $H = S_m$ and $\chi \equiv \mathbf{1}$ be the principal character.

Then $V_\chi^m(H)$ is the mth completely symmetric space over $\mathbb{C}^n$, which has dimension $\binom{m+n-1}{m}$,

and $K(T) = P_m(T)$ is the mth induced power of T.

If $A \in M_n$, then $P_m(A)$ is the mth permanental compound of A with entries equal to the permanent of

$$A[i_1, \ldots, i_m; j_1, \ldots, j_m]$$

with $1 \le i_1 \le \cdots \le i_m \le n$ and $1 \le j_1 \le \cdots \le j_m \le n$.

Operator properties of T and $K(T)$

Theorem If T has nice (algebraic or analytic) properties, say, T is normal, unitary, positive (semi-)definite, or Hermitian, then so has $K(T)$.

Question When does the converse hold?

Good cases

If $\chi \equiv \mathbf{1}$, the principal character, the converses are always valid (up to a multiple).

Suppose $\chi \equiv \varepsilon$ on S_m with $m < n$.

Then $K(T)$ is unitary/scalar if and only if T is unitary/scalar;

$\eta K(T)$ is positive definite if and only if ξT is positive definite with $\xi^m = 1$.

Bad cases

Suppose $H = S_m$, and $T = T_1 \oplus 0$ with T_1 acting on an m-dimensional subspace such that $\det(T_1) \neq 0$. Then $C_m(T)$ is the rank one normal operator with matrix representation

$$\mathrm{diag}(\det(T_1), 0, \ldots, 0).$$

So, $C_m(T)$ is Hermitian (positive semi-definite) if and only if $\det(T_1) \in \mathbb{R}$ ($\det(T_1) > 0$).

If $m = n$, then $C_m(T) = [\det(T)]$ is always normal. It is unitary if and only if $|\det(T)| = 1$; it is Hermitian if and only if .....

Results [Li and Zaharia, 2001], [Li and Tam, 2005]

A character χ is of the determinant type if $K(T) \equiv (\det T)^r I$ for some positive integer r.

A character χ is of the special type if $K(T_1 \oplus 0) \equiv (\det T_1)^r I \oplus 0$ for some positive integer r.

Theorem Suppose χ is not of the determinant type. Then $K(T)$ is nonzero normal if and only if

(i) T is normal, or

(ii) χ is of the special type, and T is unitarily similar to $T_1 \oplus 0$ with an invertible T_1.

Theorem Suppose χ is not of the determinant type. Then $K(T)$ is Hermitian (positive semi-definite, a multiple of Hermitian idempotent) if and only if

(i) ξT has the corresponding property for some $\xi^m = 1$, or

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Theorem Suppose χ is not of the determinant type.

(a) $K(T)$ is unitary if and only T is unitary.

(b) $K(T) = \eta I$ is a scalar if and only if $T = \xi I$ with $\xi^m = \eta$.

(c) $\eta K(T)$ is positive definite if and only if there is $\xi \in \mathbb{C}$ with $\xi^m = \eta$ such that ξT is positive definite.

A related problem

If $T = \xi S$ for some complex number ξ with $\xi^m = 1$, then $K(T) = K(S)$. But the converse may not be true.

Theorem Suppose χ is not of the determinant type. If $K(T) = K(S)$ is nonzero, then

(i) $T = \xi S$ with $\xi^m = 1$, or

(ii) χ is of the special type, there are unitary U and W such that $UTW = T_1 \oplus 0$ and $USW = S_1 \oplus 0$ so that $\det(T_1)^r = \det(S_1)^r$ is nonzero.

Numerical ranges and decomposable numerical ranges

The numerical range of $T : V \to V$ is defined by

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The numerical range of $T : V \to V$ is defined by

$$W(T) = \{(Tv, v) : v \in V, (v, v) = 1\}.$$

It is useful in studying T. For example,

(a) $W(T)$ is always compact and convex.

(b) $\mathrm{Sp}\,(T) \subseteq W(T)$.

(c) Conical points of $W(A)$ are reducing eigenvalues of T.

(d) $T = \mu I$ if and only if $W(T) = \{\mu\}$.

(e) $T = T^*$ if and only if $W(T) \subseteq \mathbb{R}$.

(f) T is positive semi-definite if and only if $W(T) \subseteq [0, \infty)$.

(g) T is positive definite if and only if $W(T) \subseteq (0, \infty)$.

(h) T is unitary if and only if $W(T) \subseteq \mathbf{D}$ and $\mathrm{Sp}\,(T) \subseteq \partial\mathbf{D}$, where $\mathbf{D} = \{\mu \in \mathbb{C} : |\mu| \leq 1\}$.

The decomposable numerical range of $T : V \to V$ is defined by

$$W_\chi(T) = \{(K(T)v^*, v^*) : v^* = v_1 * \cdots * v_m \text{ has unit length}\}.$$

Since not all unit vectors in $V_\chi^m(H)$ are decomposable,

$$W_\chi(T) \subseteq W(K(T))$$

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Basic properties

(a) Compactness. Yes.

(b) Convexity. No in general.

(c) $\mathrm{Sp}\,(K(T)) \subseteq W_\chi(T)$. Yes.

(d) In some cases, conical points of $W_\chi(T)$ are eigenvalues of $K(T)$.

Question Can we deduce properties of T from $W_\chi(T)$?

[Li and Zaharia, 2001]

Theorem Suppose χ is a linear irreducible character. Then

(e) $W_\chi(T) = \{\eta\}$ if and only if $T = \xi I$ with $\xi^m = \eta$;

(f) $\eta W\chi(T) \subseteq (0, \infty)$ if and only if ξT is positive definite for some ξ with $\xi^m = \eta$;

(g) $W_\chi(T) \subseteq \mathbf{D}$ and $\mathrm{Sp}\,(K(T)) \subseteq \partial\mathbf{D}$ if and only if T is unitary.

(h) $\eta W_\chi(T) \subseteq \mathbb{R}$ if and only if ξT is self-adjoint for some ξ with $\xi^m = \pm\eta$ or χ is of the special type and T is unitarily similar to $T_1 \oplus 0$ ....

The proofs used heavily the induced matrix structure of $K(A)$ for $A \in M_n$ if χ is a linear irreducible character.

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Open problem What happen for non-linear characters $\chi : H \to \mathbb{C}$?

Need some new ideas and techniques.

Inequalities

Consider the spectral norm $\|T\| = s_1(T)$,

the numerical radius $r(T) = \max\{|\mu| : \mu \in W(T)\}$,

and the spectral radius $\rho(T) = \max\{|\mu| : \mu \in \mathrm{Sp}\,(T)\}$.

Then

$$|\det(T)|^{1/n} \le \rho(T) \le r(T) \le \|T\| \le 2r(T).$$

Equality cases may lead to nice algebraic structure of T.

Theorem T is a multiple of a unitary operator if and only if

$$|\det(T)|^{1/n} = f(T)$$

for $f(T) = r(T)$ or $\|T\|$.

Question Are there similar results for induced operators?

We have

$$|\det(T)|^{m/n} \leq \rho(K(T)) \leq r_\chi(T) \leq r(K(T)) \leq \|K(T)\|.$$

[Li and Zaharia, 2001] used induced matrices to obtain results for $K_\chi(T)$ when χ is a linear irreducible character.

Theorem Suppose χ is a linear irreducible character not of the determinant type. An operator T is a multiple of a unitary operator if and only if $\rho(K(A)) = f(A)$ for $f(A) = r_\chi(A)$, $r(K(A))$ and $\|K(A)\|$.

Not many results for non-linear characters are available yet.

Preserver problems

Let $F(A)$ be a function on $A \in M_n$. We are interested in studying the structure of maps ϕ such that $F(\phi(X)) = F(X)$ for all $X \in M_n$.

(a) linear preservers of det(A) has the form

$$\phi(X) = MXN \text{ or } \phi(X) = MX^tN, \text{ det}(MN) = 1.$$

(b) linear preservers of Sp$\,(A)$ has the form

$$\phi(X) = SXS^{-1} \text{ or } \phi(X) = SX^tS^{-1}, \ S \text{ invertible.}$$

(c) linear preservers of $W(A)$ has the form

$$\phi(X) = UXU^* \text{ or } \phi(X) = UX^tU^*, \ U \text{ unitary.}$$

(d) linear preservers of $\|A\|$ has the form

$$\phi(X) = UXV \text{ or } \phi(X) = UX^tV, \; U \text{ and } V \text{ are unitary.}$$

(e) linear preservers of $r(A)$ has the form

$$\phi(X) = \mu UXU^* \text{ or } \phi(X) = \mu UXU^*, \; |\mu| = 1, \; U \text{ unitary.}$$

(f) linear preservers of $\rho(A)$, $|\det(A)|$, etc. has the form

$$\ldots$$

There are also results for multiplicative preservers of

$$\det(A), \mathrm{Sp}\,(A), W(A), \|A\|, r(A), \rho(A), |\det(A)|.$$

[Li and Zaharia, 2001] and [Cheung, Duffner, and Li, 2005] have obtained results of linear and multiplicative preservers of

$$\mathrm{Sp}\,(K(A)), W_\chi(A), \|K(A)\|, r(K(A)), r_\chi(A), \rho(K(A))$$

for linear irreducible characters. The preservers are often just ξ multiple of the standard form with $\xi^m = 1$.

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There are many other challenging / interesting problems. Any help would be welcome!

Thank you for your attention!