Posing questions, solving problems, and doing research in mathematics

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Why are teaching and learning mathematics not interesting?

* Mathematics is difficult.
* Mathematics is boring.
* Mathematics is useless.
* Mathematics is too abstract.
* There are too many formulas.
* There are too many uninspiring artificial exercises.

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But

* Mathematics is beautiful.
* Mathematics is useful.
* Study and research in mathematics is very creative.

Questions

* How to change the inaccurate perspectives?
* How to make teaching and learning mathematics enjoyable?
* How to arouse the interest of teachers and students?
* How to increase the confidence of teachers and students?

A proposed scheme

Find / devise problems that interested teachers and students, whose study require deep understanding of current knowledge and exploring new techniques.

Emphases

* Teachers should use problems that excite themselves and likely to excite students.
  (vs. using "interesting" problems imposed on them).
* The problems should engage students of all levels.
* The problems should help students learn a certain topic or concept effectively.
* The problems could show connections of mathematics to other areas.

Do such questions exist?

Yes! Teachers may carefully adapt existing material.
Teachers may devise the problems themselves.
Teachers may encourage students to generate them.

Example What can you say about $1 + 2 = 3$?

* Is it true that $1 + 2 + 3 = 4$? How about $1 + \cdots + 5 = 6$?
* Why is it not possible to have $1 + \cdots + n = n + 1$?
  Here students are asked to do proofs!
  (without even realizing that!)
* How about $1 + \cdots + k = (k + 1) + \cdots + n$?
  Set up a table, use computer, use the formula for adding consecutive numbers ...

How about the following easier/harder problems?

* Can we divide $1, 2, 3, 4$ into two groups so that each group has the same sum?

  Here students are asked to practice additions (without knowing that)!

* How about $1, \ldots, n$?

  One has to write down the general construction for a proof.

* If it can be done, how many ways can you do it?

* How about dividing the numbers into 3 groups, k groups, etc.

An observation of a problem solving class

* Use a paper to form a cylinder (without top and bottom).
* Cut and paste to get a paper with half the height and double the width to form another cylinder.
* Cut and paste to get a paper with 1/3 of the height and 3 times the width to form yet another cylinder.

Students are then divided into small groups to study the following.

Question Which cylinder has a larger volume?

Answers

* All cylinders have the same volume.
* By experience, the shortest one holds more stuff.
* By experiments, the shortest one holds more stuff.
* Use some real or fake numbers to show that the shortest one holds more.
* Use formula to show that

$$\pi(3r)^2(h/3) \geq \pi(2r)^2(h/2) \geq \pi r^2 h.$$

Other interesting comments and ideas

* The shorter one always has the larger volume.

* What if the height go to zero?

* What if we do a square or irregular base prism?

* What if we include the top and bottom?

Largest convex polygons with given sides

* Given lengths l_1, l_2, l_3, when could we form a triangle (with maximum area)?

* Given lengths $l_1, \ldots, l_4$, when could we form a convex quadrilateral (with maximum area)?

* Given lengths $l_1, \ldots, l_n$, when could we form an n-sided convex polygon (with maximum area)?

* Can we rearrange the order of the sides?

* How about the same problem on the sphere, on the hyperbolic plane?

Challenges

* Can you find / devise a problem?
* Can you find / devise a problem that interest you and your students?
* Can you find a problem that will allow students with different background to enjoy?
* Can you engage your students to ask (interesting) questions?
* If the problem is too hard, can you consider a simpler version?
* If the problem is too easy, can you consider a harder one?
* You may always get help from your professors.

* It is good if you identify a “Gauss” in your class.

* It is even better if you change the attitudes of many people towards mathematics.

Thank you for your attention!