Posing questions, solving problems, and doing research in mathematics

Chi-Kwong Li
Department of Mathematics
College of William and Mary
Williamsburg, Virginia 23187-8795
Why are teaching and learning mathematics not interesting?

* Mathematics is difficult.
* Mathematics is boring.
* Mathematics is useless.
* Mathematics is too abstract.
* There are too many formulas.
* There are too many uninspiring artificial exercises.

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But

* Mathematics is beautiful.

* Mathematics is useful.

* Study and research in mathematics is very creative.

Questions

* How to change the inaccurate perspectives?

* How to make teaching and learning mathematics enjoyable?

* How to arouse the interest of teachers and students?

* How to increase the confidence of teachers and students?
A proposed scheme

Find / devise problems that interested teachers and students, whose study require deep understanding of current knowledge and exploring new techniques.

Emphases

* Teachers should use problems that excite themselves and likely to excite students. (vs. using “interesting” problems imposed on them).
* The problems should engage students of all levels.
* The problems should help students learn a certain topic or concept effectively.
* The problems could show connections of mathematics to other areas.
Do such questions exist?

Yes! Teachers may carefully adapt existing material.

- Teachers may devise the problems themselves.
- Teachers may encourage students to generate them.

**Example** What can you say about $1 + 2 = 3$?

* Is it true that $1 + 2 + 3 = 4$? How about $1 + \cdots + 5 = 6$?

* Why is it not possible to have $1 + \cdots + n = n + 1$?
  
  Here students are asked to do proofs!
  
  (without even realizing that!)

* How about $1 + \cdots + k = (k + 1) + \cdots + n$?
  
  Set up a table, use computer, use the formula for adding consecutive numbers ...
How about the following easier/harder problems?

* Can we divide 1, 2, 3, 4 into two groups so that each group has the same sum?
  Here students are asked to practice additions (without knowing that)!

* How about 1, . . . , n?
  One has to write down the general construction for a proof.

* If it can be done, how many ways can you do it?

* How about dividing the numbers into 3 groups, $k$ groups, etc.
An observation of a problem solving class

* Use a paper to form a cylinder (without top and bottom).

* Cut and paste to get a paper with half the height and double the width to form another cylinder.

* Cut and paste to get a paper with $1/3$ of the height and $3$ times the width to form yet another cylinder.

Students are then divided into small groups to study the following.

**Question** Which cylinder has a larger volume?
Answers

* All cylinders have the same volume.

* By experience, the shortest one holds more stuff.

* By experiments, the shortest one holds more stuff.

* Use some real or fake numbers to show that the shortest one holds more.

* Use formula to show that

\[ \pi (3r)^2 \left( \frac{h}{3} \right) \geq \pi (2r)^2 \left( \frac{h}{2} \right) \geq \pi r^2 h. \]
Other interesting comments and ideas

* The shorter one always has the larger volume.

* What if the height go to zero?

* What if we do a square or irregular base prism?

* What if we include the top and bottom?
Largest convex polygons with given sides

* Given lengths $l_1, l_2, l_3$, when could we form a triangle (with maximum area)?

* Given lengths $l_1, \ldots, l_4$, when could we form a convex quadrilateral (with maximum area)?

* Given lengths $l_1, \ldots, l_n$, when could we form an $n$-sided convex polygon (with maximum area)?

* Can we rearrange the order of the sides?

* How about the same problem on the sphere, on the hyperbolic plane?
Challenges

* Can you find / devise a problem?

* Can you find / devise a problem that interest you and your students?

* Can you find a problem that will allow students with different background to enjoy?

* Can you engage your students to ask (interesting) questions?

* If the problem is too hard, can you consider a simpler version?

* If the problem is too easy, can you consider a harder one?

* You may always get help from your professors.
* It is good if you identify a “Gauss” in your class.

* It is even better if you change the attitudes of many people towards mathematics.
Thank you for your attention!