Eigenvalues and singular values inequalities: Some recent results

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Introduction

H_n: the set of $n \times n$ Hermitian matrices.

Denote the eigenvalues of $A \in H_n$ by

$$\lambda_1(A) \geq \cdots \geq \lambda_n(A).$$

Theorem [Weyl] Let $A, E \in H_n$. If $1 \leq j \leq n$, then

$$\lambda_n(E) \leq \lambda_j(A+E) - \lambda_j(A) \leq \lambda_1(E).$$

Theorem [Liskii] Let $A, E \in H_n$. If $1 \leq j_1 < \cdots < j_m \leq n$, then

$$\sum_{t=1}^{m} \lambda_{n-t+1}(E) \leq \sum_{t=1}^{m} [\lambda_{j_t}(A+E) - \lambda_{j_t}(A)] \leq \sum_{t=1}^{m} \lambda_t(E).$$

A simple proof of Liskii's result [Li and Mathias, 1999].

Step 1 [max-min characterization of eigenvalues]
If $G_{n,k}$ = the set of all k-dimensional subspace in $\mathbb{C}^n$, then

$$\lambda_k(A) = \max_{V \in G_{n,k}} \min\{x^*Ax : x \in V,\ x^*x = 1\}.$$

Proof. Let $A = \sum_{j=1}^n \lambda_j(A) u_j u_j^*$, the spectral decomposition.

If $V \in G_{n,k}$, then $V \cap \operatorname{span}\{u_k, \ldots, u_n\} \neq 0$.
So, V contains a unit vector $y = \sum_{j=k}^n c_j u_j$, and

$$y^*Ay = \textstyle\sum_{j=k}^n c_j^2 \lambda_j(A) \le \lambda_k(A).$$

Let $V = \operatorname{span}\{u_1, \ldots, u_k\}$. If $x = \sum_{j=1}^k d_j u_j \in V$ is a unit vector, then

$$x^*Ax = \textstyle\sum_{j=1}^k d_j^2 \lambda_j(A) \ge \lambda_k(A) = u_k^* A u_k.$$

Step 2 If F is positive semidefinite, then

$$\lambda_k(A+F) \geq \lambda_k(A), \qquad j=1,\ldots,n.$$

Proof. For each k,

$$\begin{aligned}
\lambda_k(A) &= \max_{V\in G_{n,k}} \min\{x^*Ax : x\in V,\ x^*x=1\} \\
&\leq \max_{V\in G_{n,k}} \min\{x^*(A+F)x : x\in V,\ x^*x=1\} \\
&= \lambda_k(A+E).
\end{aligned}$$

Step 3 Suppose $1 \leq j_1 < \cdots < j_m \leq n$ are given.

Assume $\lambda_m(E) = 0$. Else, replace (A, E) by $(A - \gamma I, E - \gamma I)$ with $\gamma = \lambda_m(E)$ so that both sides of

$$\textstyle\sum_{t=1}^{m} [\lambda_{j_t}(A + E) - \lambda_{j_t}(A)] \leq \sum_{t=1}^{m} \lambda_t(E)$$

are reduced by $m\lambda_m(E)$.

If $E = \sum_{j=1}^{n} \lambda_j(E) f_j f_j^*$, set $E_+ = \sum_{j=1}^{n} \lambda_j(E) f_j f_j^*$,

then $E_+ - E$ and E_+ are positive semidefinite so that

$$\sum_{t=1}^{m} [\lambda_{j_t}(A + E) - \lambda_{j_t}(A)] \leq \sum_{t=1}^{m} [\lambda_{j_t}(A + E_+) - \lambda_{j_t}(A)]$$

$$\leq \sum_{t=1}^{n} [\lambda_t(A + E_+) - \lambda_t(A)] = \operatorname{tr} E_+ = \sum_{j=1}^{m} \lambda_j(E).$$

Perturbation of block Hermitian matrices

Let $A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix}$ have eigenvalues $\lambda_1 \geq \cdots \geq \lambda_k$,

$\widetilde{A} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}$ have eigenvalues $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_k$.

Suppose

$\|E\|$ is the spectral norm of the matrix E,

$\mathrm{Sp}\,(X)$ is the spectrum of the square matrix X,

$$\begin{aligned}\eta &= \min\{|\mu_1 - \mu_2| : \mu_1 \in \mathrm{Sp}\,(H_1), \mu_2 \in \mathrm{Sp}\,(H_2)\} \\ &= \text{the spectral gap between the spectra of } H_1 \text{ and } H_2.\end{aligned}$$

There are two kinds of perturbation bounds on $|\lambda_i - \widetilde{\lambda}_i|$:

1. $|\lambda_i - \widetilde{\lambda}_i| \leq \|E\|$, and
2. $|\lambda_i - \widetilde{\lambda}_i| \leq \|E\|^2/\eta$
   if there is gap η between $\mathrm{Sp}\,(H_1)$ and $\mathrm{Sp}\,(H_2)$.

The first bound overestimates the changes when $\|E\| \ll \eta$. The second bound blows up when η is small.

Our **goal** is to show that for $i = 1, \ldots, k$,

$$|\lambda_i - \widetilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}},$$

and obtain similar bounds for singular values of matrices under block perturbations.

Theorem *Consider the Hermitian matrices*

$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \quad \textit{have eigenvalues} \quad \lambda_1 \geq \cdots \geq \lambda_k, \textit{ and}$$

$$\widetilde{A} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \quad \textit{have eigenvalues} \quad \widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_k.$$

where $H_1 \in M_m,\ H_2 \in M_n$. *For* $i = 1, \ldots, m+n$,

$$|\lambda_i - \widetilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}},$$

where

$$\eta_i = \begin{cases} \min\{|\widetilde{\lambda}_i - \mu| : \mu \in \mathrm{Sp}\,(H_1)\} & \textit{if } \widetilde{\lambda}_i \in \mathrm{Sp}\,(H_2), \\ \min\{|\widetilde{\lambda}_i - \mu| : \mu \in \mathrm{Sp}\,(H_2)\} & \textit{if } \widetilde{\lambda}_i \in \mathrm{Sp}\,(H_1). \end{cases}$$

Ideas of proof.

For $k = 1$ or $m + n$, use a clever Schur complement argument and some calculus.

For $1 < k < m + n$. Use induction on a judicious choice of compression with interlacing inequalities.

Details of proofs.

- * Replace A by $(U_1 \oplus U_2)^* A (U_1 \oplus U_2)$ by suitable $U_1 \oplus U_2$ so that H_1 and H_2 are diagonal matrices with diagonal entries arranged in descending order.
- * Perturb the diagonal entries and assume that they are distinct; treat the general case by continuity.

If $m + n = 2$, direct computation; the matrix $A = \begin{pmatrix} \alpha & \epsilon \\ \epsilon & \beta \end{pmatrix}$ has eigenvalues

$$\lambda_\pm = \tfrac{\alpha+\beta \pm \sqrt{(\alpha+\beta)^2 - 4(\alpha\beta - \epsilon^2)}}{2} = \tfrac{\alpha+\beta \pm \sqrt{(\alpha-\beta)^2 + 4\epsilon^2}}{2}.$$

Assume that $m + n > 2$, and the result is true for Hermitian matrices of size at most $m + n - 1$.

Suppose $A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix}$. Then $\lambda_1 I - A$ has zero as the smallest eigenvalue. Assume $\tilde{\lambda}_1$ occurs in H_1. By Schur complement, $\lambda_1 I - A$ is congruent to

$$\left[(\lambda_1 I_m - H_1) - E^*(\lambda_1 I_n - H_2)^{-1}E\right] \oplus (\lambda_1 I_n - H_2)$$

has zero as the smallest eigenvalue. So,

$$\delta_1 = \lambda_1 - \tilde{\lambda}_1 = \lambda_m(\lambda_1 I_m - H_1) \le \lambda_1(E^*(\lambda_1 I_n - H_2)^{-1}E).$$

Hence

$$\delta_1 \le \lambda_1(E^*(\lambda_1 I_n - H_2)^{-1}E) \le \|E\|^2/(\delta_1 + \eta_1).$$

and

$$\delta_1 \le \frac{2\|E\|}{\eta_1 + \{\eta_1^2 + 4\|E\|^2\}^{1/2}}.$$

Next, suppose $1 < k < m+n$. The result trivially holds if $\lambda_i = \widetilde{\lambda}_i$. So, assume that $\widetilde{\lambda}_i > \lambda_i$.
Otherwise, replace $(A, \widetilde{A}, i)$ by $(-A, -\widetilde{A}, m+n-i+1)$.
Delete the row and column of A that contain the diagonal entry $\widetilde{\lambda}_n$ to get

$$\widehat{A} = \begin{pmatrix} \widehat{H}_1 & \widehat{E}^* \\ \widehat{E} & \widehat{H}_2 \end{pmatrix}$$

with eigenvalues $\nu_1 \geq \cdots \geq \nu_{m+n-1}$. By the interlacing inequalities $\lambda_i \geq \nu_i$ and hence

$$\widetilde{\lambda}_i - \lambda_i \leq \widetilde{\lambda}_i - \nu_i. \tag{1}$$

Now, $\widetilde{\lambda}_i$ is the ith largest diagonal entries in $\widehat{A}$ and

$$\widehat{\eta}_i \geq \eta_i.$$

Also, note that $\|\widehat{E}\| \le \|E\|$. Thus,

$$\begin{aligned}
|\lambda_i - \widetilde{\lambda}_i| &= \widetilde{\lambda}_i - \lambda_i && \text{because } \widetilde{\lambda}_i > \lambda_i \\
&\le \widetilde{\lambda}_i - \nu_i && \text{by (1)} \\
&\le \frac{2\|\widehat{E}\|^2}{\widehat{\eta}_i + \sqrt{\widehat{\eta}_i^2 + 4\|\widehat{E}\|^2}} && \text{by induction assumption} \\
&\le \frac{2\|\widehat{E}\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|\widehat{E}\|^2}} && \text{because } \widehat{\eta}_i \ge \eta_i \\
&= \frac{1}{2}\sqrt{\eta_i^2 + 4\|\widehat{E}\|^2} - \eta_i \\
&\le \frac{1}{2}\sqrt{\eta_i^2 + 4\|E\|^2} - \eta_i && \text{because } \|\widehat{E}\| \le \|E\| \\
&= \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}}. && \blacksquare
\end{aligned}$$

Singular values

Denote the sequence of singular values of a complex $p \times q$ matrix X by $\sigma(X) = (\sigma_1(X), \ldots, \sigma_k(X))$, where $k = \max\{p, q\}$.

Theorem Let

$B = \begin{pmatrix} G_1 & E_1 \\ E_2 & G_2 \end{pmatrix}$ and $\widetilde{B} = \begin{pmatrix} G_1 & O \\ O & G_2 \end{pmatrix}$ have singular values

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ and $\widetilde{\sigma}_1 \geq \widetilde{\sigma}_2 \geq \cdots \geq \widetilde{\sigma}_r$, respectively,

where $G_1 \in M_{m,k}$, $G_2 \in M_{n,\ell}$ are non-trivial,
and $r = \max\{m + n, k + \ell\}$.

Define $\epsilon = \max\{\|E_1\|, \|E_2\|\}$, and

$$\eta_i = \begin{cases} \min\limits_{\mu \in \sigma(G_2)} |\widetilde{\sigma}_i - \mu| & \text{if } \widetilde{\sigma}_i \in \sigma(G_1), \\ \min\limits_{\mu \in \sigma(G_1)} |\widetilde{\sigma}_i - \mu| & \text{if } \widetilde{\sigma}_i \in \sigma(G_2), \end{cases}$$

and $\qquad \eta = \min_{1 \leq i \leq m+n} \eta_i.$

Then for $i = 1, 2, \cdots, \min\{m+n, k+\ell\}$, we have

$$|\sigma_i - \widetilde{\sigma}_i| \le \frac{2\epsilon^2}{\eta_i + \sqrt{\eta_i^2 + 4\epsilon^2}} \le \frac{2\epsilon^2}{\eta + \sqrt{\eta^2 + 4\epsilon^2}},$$

and $\sigma_i = \widetilde{\sigma}_i = 0$ for $i > \min\{m+n, k+\ell\}$.

Theorem Suppose $B = (\, G \; E \,)$ and $\tilde{B} = (\, G \; O \,)$ are $p \times q$ matrices with singular values

$$\sigma_1 \ge \cdots \ge \sigma_{\max\{p,q\}} \quad \text{and} \quad \widetilde{\sigma}_1 \ge \ldots \ge \widetilde{\sigma}_{\max\{p,q\}},$$

respectively. Then for $i = 1, \ldots, \min\{p, q\}$,

$$|\sigma_i - \widetilde{\sigma}_i| \le \frac{2\|E\|^2}{2\widetilde{\sigma}_i + \sqrt{\widetilde{\sigma}_i^2 + 4\|E\|^2}}.$$

Thank you for your attention!