Recent study in preserver problems

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1. Preserver Problems on Matrices/Operators

Let $\mathcal{M}$ be a vector space or algebras of matrices or operators.

Characterize $\phi : \mathcal{M} \to \mathcal{M}$ with some special properties:

(a) $f(\phi(A)) = f(A)$ for all $A \in \mathcal{M}$, where f is a given function on $\mathcal{M}$;

(b) $\phi(\mathcal{S}) \subseteq \mathcal{S}$ or $\phi(\mathcal{S}) = \mathcal{S}$ for a certain subset $\mathcal{S} \subseteq \mathcal{M}$;

(c) $\phi(A) \sim \phi(B)$ in $\mathcal{M}$ whenever $A \sim B$ in $\mathcal{M}$ for a certain relations $\sim$ on $\mathcal{M}$.

Very often, ϕ is assumed to be linear, additive, multiplicative, analytic, injective, surjective, unital ....

Examples of preservers of functions

Theorem [Frobenius, 1897] A linear operator $\phi : M_n \to M_n$ satisfies

$$\det(A) = \det(\phi(A)) \text{ for all } A \in M_n$$

if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that ϕ has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN.$$

Theorem [Marcus and Purves, 1959] Let Eig (A) be the set of of eigenvalues of $A \in M_n$. A linear map $\phi : M_n \to M_n$ satisfies

$$\text{Eig}\,(\phi(A)) = \text{Eig}\,(A) \text{ for all } A \in M_n$$

if and only if there is an invertible matrix $S \in M_n$ such that ϕ has the form

$$A \mapsto S^{-1}AS \quad \text{or} \quad A \mapsto S^{-1}A^tS.$$

Examples of preservers of sets

Theorem [Dieudonné, 1949] A *bijective* linear map $\phi : M_n \to M_n$ mapping *the set of singular matrices* into itself has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN$$

for some $M, N \in M_n$ with $\det(MN) \neq 0$.

Theorem [Marcus & Purves, 1959] A linear map $\phi : M_n \to M_n$ mapping *the set of invertible matrices* into itself has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN$$

for some $M, N \in M_n$ with $\det(MN) \neq 0$.

Examples of preservers of relations

Theorem [Hua, 1951] A bijective map $\phi : M_n \to M_n$ satisfies the condition that

$$\operatorname{rank}(\phi(A) - \phi(B)) = 1 \text{ if and only if } \operatorname{rank}(A - B) = 1$$

if and only if there are $M, N, R \in M_n$ with $\det(MN) \neq 0$ such that ϕ has the form

$$A \mapsto MAN + R \quad \text{or} \quad A \mapsto MA^tN + R.$$

Theorem [Hiai, 1987] A linear map $\phi : M_n \to M_n$ satisfies the condition that

$$\phi(A) \text{ is similar to } \phi(B) \text{ whenever } A \text{ is similar to } B$$

if and only if there is a fixed $B \in M_n$ such that ϕ has the form

$$A \mapsto (\operatorname{tr} A)B,$$

or there are $a, b \in \mathbb{C}$ and an invertible S such that ϕ has the form

$$A \mapsto aS^{-1}AS + b(\operatorname{tr} A)I$$

or

$$A \mapsto aS^{-1}A^tS + b(\operatorname{tr} A)I.$$

Some recent results on spectra preserving maps

Theorem [Marcus & Purves, 1959], [Jafarian & Sourour, 1986] Linear preservers of eigenvalues or spectra of matrices have the standard form $A \mapsto S^{-1}AS$ or $A \mapsto S^{-1}A^tS$.

Theorem [Omladič and Šemrl, 1991] Additive preservers of spectrum on M_n are linear.

Theorem [Hochwald, 1994], [Cheung, Fallat, and Li, 2002] Multiplicative preservers of eigenvalues or spectra of matrices have the standard from.

Theorem [Baribeau and Ransford, 2000]
Continuously differentiable preservers of spectra on M_n are local (anti-)automorphisms.

$$A \mapsto S_A^{-1}AS_A \text{ or } A \mapsto S_A^{-1}A^tS_A.$$

A new direction

Theorem [Bhatia, Šemrl, Sourour, 1999]
A mapping $\phi : M_n \to M_n$ satisfies

$$\mathrm{Sp}\,(A - B) = \mathrm{Sp}\,(\phi(A) - \phi(B)) \text{ for all } A, B \in M_n$$

is real affine. So, the mapping has the form

$$A \mapsto S^{-1}AS + R \text{ or } A \mapsto S^{-1}A^tS + R.$$

Theorem [Molnár, 2001] A surjective mapping $\phi : M_n \to M_n$ satisfies

$$\mathrm{Sp}\,(AB) = \mathrm{Sp}\,(\phi(A)\phi(B)) \text{ for all } A, B \in M_n$$

has the form

$$A \mapsto \pm S^{-1}AS \text{ or } A \mapsto \pm S^{-1}A^tS.$$

Theorem [Chan, Li, Sze, 2005] Let $A * B = AB$, ABA, or $AB + BA$, and let $\mathcal{M} = M_n$ or H_n. A mapping $\phi : \mathcal{M} \to \mathcal{M}$ satisfies

$$\mathrm{Sp}\,(A * B) = \mathrm{Sp}\,(\phi(A) * \phi(B)) \text{ for all } A, B \in \mathcal{M}$$

has the form

$$A \mapsto \mu S^{-1} A S \text{ or } A \mapsto \mu S^{-1} A^t S.$$

Ideas of proof.

* If ϕ preserves eigenvalues, then it satisfies

$$\mathrm{tr}\,(\phi(A)\phi(B)) = \mathrm{tr}\,(AB) \text{ for all } A, B.$$

  Then ϕ is invertible linear, and has the standard form.
* Consider the restriction of ϕ on the dense subset $\mathcal{S}$ of matrices with distinct eigenvalues. Then ϕ equals a linear map L_A on each neighborhood of a matrix $A \in \mathcal{S}$.
* Then show that $L_A = L_B$ for each pair of $A, B \in \mathcal{S}$. So, $\phi = L$ on $\mathcal{S}$.
* Now, use algebraic arguments to show that $\phi = L$ on the whole matrix space. (Note that no continuity assumption.)

Further research

There are many other interesting problems.

For example, study the preservers of spectral radius, spectral norm, numerical radius, numerical range, etc.

One may consider other domain and co-domains such as triangular matrices, matrices over other rings or semi-rings, etc.

One may also consider $\phi : \mathcal{M} \to \mathcal{M}'$ for different $\mathcal{M}$ and $\mathcal{M}'$.

An invitation

You are most welcome to join the club of preserverists!

Thank you for your attention!