

# Linear Operators on Matrix Algebras that Preserve the Numerical Range, Numerical Radius or the States

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## 1. Classical Result

Linear preserver problems concern the characterization of linear maps on matrices or operators that leave invariant a certain function, subset, or relation. For example,

**Theorem** [Frobenius, 1897] A linear operator  $\phi : M_n \rightarrow M_n$  satisfies

$$\det(\phi(A)) = \det(A) \text{ for all } A \in M_n$$

if and only if there are  $M, N \in M_n$  with  $\det(MN) = 1$  such that  $\phi$  has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^t N.$$

Let  $A \in M_n$ . The classical numerical range is

$$W(A) = \{x^* Ax : x \in \mathbb{C}^n, x^* x = 1\}.$$

**Theorem** [Pellegrini, 1975] A linear map  $\phi : M_n \rightarrow M_n$  satisfies

$$W(\phi(A)) = W(A) \quad \text{for all } A \in M_n$$

if and only if there is a unitary matrix  $U$  such that  $\phi$  has the form

$$A \mapsto U^* A U \quad \text{or} \quad A \mapsto U^* A^t U.$$

## 2. States, numerical ranges and radii

Define the dual norm of a norm  $\nu$  on  $\mathbb{C}^n$  by

$$\nu^D(y) = \max\{|(y, x)| : \nu(x) \leq 1\}.$$

Then define the set of *vector states* on  $M_n$  with respect to  $\nu$  is

$$\mathcal{R} = \{xy^* : x, y \in \mathbb{C}^n, \nu^D(x) = \nu(y) = x^*y = 1\},$$

and the *spatial numerical range* of  $A \in M_n$  by

$$\begin{aligned} W(A) &= W_\nu(A) = \{(A, Z) : Z \in \mathcal{R}\} \\ &= \{x^*Ay : x, y \in \mathbb{C}^n, \nu^D(x) = \nu(y) = x^*y = 1\}. \end{aligned}$$

Consider the operator norm on  $M_n$  induced by  $\nu$ :

$$\|A\| = \max\{\nu(Ax) : x \in \mathbb{C}^n, \nu(x) \leq 1\}.$$

The dual norm of  $\|\cdot\|$  on  $M_n$  is defined by

$$\|B\|^D = \{|(B, A)| : \|A\| \leq 1\}.$$

Define the set of *states* on  $M_n$  with respect to the norm  $\|\cdot\|$  by

$$\mathcal{S} = \{B \in M_n : \text{tr } B = \|B\|^D = 1\},$$

and the *algebra numerical range* by

$$\begin{aligned} V(A) &= V_\nu(A) = \{(A, Z) : Z \in \mathcal{S}\} \\ &= \{(A, B) : B \in M_n, \text{tr } B = \|B\|^D = 1\}. \end{aligned}$$

### Example

If  $p, q \in [1, \infty]$  and  $1/p + 1/q = 1$ , then  $\ell_p$ -norm and  $\ell_q$ -norm are dual to each other;

$$\mathcal{R} = \{xy^* : \ell_q(x) = 1 = \ell_p(y), y^*x = 1\},$$

$$W(A) = \{x^*Ay : \ell_q(x) = 1 = \ell_p(y), y^*x = 1\}.$$

Except for  $p, q \in \{1, \infty\}$ , all matrices in  $\mathcal{R}$  are permutationally similar to  $R \oplus 0_{n-k}$  for some  $k \in \{0, \dots, n-1\}$  where  $R$  has no zero entries.

In the exceptional case, if  $\nu = \ell_\infty$  and

$$\mathcal{E}_i = \{e_i(v_1, \dots, v_n) : 1 = v_i = |v_j| \text{ for all } j\},$$

then

$$\cup_{i=1}^n \mathcal{E}_i \subseteq \mathcal{R} \subseteq \mathcal{S} = \text{conv} (\cup_{i=1}^n \mathcal{E}_i).$$

### 3. Preservers of states and numerical ranges

Let  $GP(n)$  be the group of generalized permutation matrices in  $M_n$ . If  $\nu$  satisfies  $\nu(Px)$  for any  $P \in GP(n)$ , then  $\nu$  is *symmetric norm*. For example, all the  $\ell_p$ -norms are symmetric norms.

**Theorem 1** Let  $\nu$  be a symmetric norm not equal to a multiple of the  $\ell_q$  norm with  $q \in \{1, 2, \infty\}$ . The following conditions are equivalent for a linear operator  $\phi$  on  $M_n$ .

- (a)  $W(\phi(A)) = W(A)$  for all  $A \in M_n$ .
- (b)  $V(\phi(A)) = V(A)$  for all  $A \in M_n$ .
- (c) There exists  $Q \in GP(n)$  such that

$$\phi(A) = Q^* A Q \quad \forall A \in M_n.$$

**Theorem 2** Let  $\nu$  be a symmetric norm not equal to a multiple of the  $\ell_q$  norm with  $q \in \{1, 2, \infty\}$ . The following conditions are equivalent for a linear operator  $\phi$  on  $M_n$ .

- (a)  $\phi(\mathcal{R}) = \mathcal{R}$ .
- (b)  $\phi(\mathcal{S}) = \mathcal{S}$ .
- (c) There exists  $Q \in GP(n)$  such that

$$\phi(A) = QAQ^* \quad \forall A \in M_n.$$

### Facts

1.  $V(A)$  is the convex hull of  $W(A)$ .
2.  $\phi$  preserves  $\mathcal{R}$  if and only if  $\phi^*$  preserve  $W(A)$ .
3.  $\phi$  preserves  $\mathcal{S}$  if and only if  $\phi^*$  preserve  $V(A)$ .

By these facts we can focus on the proof of Theorem 2.

## Ideas of proof.

First, show that (c)  $\Rightarrow$  (a)  $\Rightarrow$  (b).

Here we use the fact that  $\mathcal{S}$  is the convex hull of  $\mathcal{R}$ .

Assume that  $\phi(\mathcal{S}) = \mathcal{S}$ .

Step 1. Show that  $\phi$  preserves the inner product.

Step 2. Show that  $\phi$  preserves diagonal matrices.

Step 3. Show that  $\phi(\mathcal{C}) = \mathcal{C}$ , where

$$\mathcal{C} = \{D e e^t D^* : D \in GP(n)\}.$$

Step 4. Use the result on rank one correlation matrix preservers in [LP,2003] and some arguments to finish the proof.

## 4. Numerical radius preservers

The *norm numerical radius*

$$r(A) = \max\{|\mu| : \mu \in W(A)\} = \max\{|\mu| : \mu \in V(A)\}.$$

**Theorem 3** A linear operator  $\phi$  on  $M_n$  satisfies

$$r(\phi(A)) = r(A) \quad \text{for all } A \in M_n$$

if and only if there is a complex number  $\mu$  of modulus 1 such that

$$W(\mu\phi(A)) = W(A) \quad \text{for all } A \in M_n.$$

In other words,  $\phi$  preserves the numerical radius if and only if it is a unit multiple of a numerical range preserver.

A key step in the proof is proving the following.

**Proposition** Let  $\mathcal{L}$  be the set of matrices  $A \in M_n$  such that for every  $Y \in M_n$  there exists a complex unit  $\eta$  such that

$$r(\eta A + Y) = 1 + r(Y).$$

Then  $A \in \mathcal{L}$  if and only if  $A = \mu I$  for some  $\mu \in \mathbb{C}$  with  $|\mu| = 1$ .

## 5. Exceptional cases

**Theorem 4** Let  $W(A)$  be induced by the  $\ell_1$  or the  $\ell_\infty$  norm on  $\mathbb{C}^n$ . Then the following conditions are equivalent for a linear operator  $\phi$  on  $M_n$ .

- (a)  $W(\phi(A)) = W(A)$  for all  $A \in M_n$ .
- (b) There exists  $Q \in GP(n)$  such that

$$\phi(A) = Q^* A Q \quad \forall A \in M_n,$$

or equivalently,

$$\phi^*(A) = Q A Q^* \quad \forall A \in M_n.$$

- (c)  $\phi^*(\mathcal{R}) = \mathcal{R}$ .

**Theorem 5** Let  $V(A)$  be induced by the  $\ell_1$  or the  $\ell_\infty$  norm on  $\mathbb{C}^n$ . Then the following conditions are equivalent for a linear operator  $\phi$  on  $M_n$ .

- (a)  $V(\phi(A)) = V(A)$  for all  $A \in M_n$ .
- (b) There exist  $P, Q_1, \dots, Q_n \in GP(n)$  with  $Q_j e_j = e_j$  for  $j \in \{1, \dots, n\}$  such that for all  $A = [A_1 | \dots | A_n] \in M_n$ ,

$$\phi([A_1 | \dots | A_n]) = P^*[Q_1 A_1 | \dots | Q_n A_n]P$$

or equivalently

$$\phi^*([A_1 | \dots | A_n]) = P[\tilde{Q}_1 A_1 | \dots | \tilde{Q}_n A_n]P^*$$

with  $\tilde{Q}_j = P^*Q_{\pi(j)}P$  where  $\pi$  is the permutation determined by  $Pe_j = e_{\pi(j)}$ .

- (c)  $\phi^*(\mathcal{S}) = \mathcal{S}$ .

The next result shows that the group of numerical radius isometries are much larger than the unit multiples of the numerical range preservers.

**Theorem 6** Suppose  $r = r_1$  is the norm numerical radius associated with the  $\ell_1$  norm on  $\mathbb{C}^n$ . Then a linear operator  $\phi$  on  $M_n$  satisfies

$$r(\phi(A)) = r(A) \quad \text{for all } A \in M_n$$

if and only if there exist  $P, Q_1, \dots, Q_n \in GP(n)$  such that  $\phi$  has the form

$$[A_1 | \cdots | A_n] \mapsto [Q_1 A_1 | \cdots | Q_n A_n] P.$$

## 6. Further research

1. Extend the results to triangular block matrices.
2. Extend the results to  $B(H)$  for a separable  $H$ .
3. Extend the results to general Banach algebras.
4. Consider other classes of  $\nu$  such as the class of absolute norms or the class of permutation invariant norms.

THANK YOU FOR YOUR ATTENTION!