

**Eigenvalues, singular values,
and Littlewood-Richardson Coefficients**

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1. Sum of Hermitian (Real Symmetric) Matrices

Let \mathbf{H}_n be the set of $n \times n$ Hermitian matrices, and let $X \in \mathbf{H}_n$. Denote by

$$\lambda(X) = (\lambda_1(X), \dots, \lambda_n(X)) \quad \text{with} \quad \lambda_1(X) \geq \dots \geq \lambda_n(X)$$

the vector of eigenvalues of X .

Problem Let $A, E \in \mathbf{H}_n$ and $\tilde{A} = A + E$. Determine inequalities relating the eigenvalues of \tilde{A} and those of A and E .

Some well known inequalities

[Weyl] $\lambda_n(E) \leq \lambda_i(\tilde{A}) - \lambda_i(A) \leq \lambda_1(E)$.

[Liskii-Mirsky-Wielandt] Suppose $1 \leq i_1 < \dots < i_m \leq n$ and $1 \leq j_1 < \dots < j_m \leq n$. Then

$$\sum_{s=1}^m \lambda_{n-s+1}(E) \leq \sum_{s=1}^m (\lambda_{j_s}(\tilde{A}) - \lambda_{j_s}(A)) \leq \sum_{s=1}^m \lambda_s(E).$$

Consequently, for any unitarily invariant norm $\|\cdot\|$ we have

$$\|\tilde{A} - A\| \leq \|E\|.$$

[Freede and Thompson] Suppose $1 \leq i_1 < \cdots < i_m \leq n$ and $1 \leq j_1 < \cdots < j_m \leq n$. If $(i_m + j_m - m \leq n)$, then

$$\sum_{s=1}^m \lambda_{i_s + j_s - s}(\tilde{A}) \leq \sum_{s=1}^m \lambda_{i_s}(A) + \sum_{s=1}^m \lambda_{j_s}(E).$$

The complete answer

Theorem [Klyaschko 1998, Fulton 1998] There exist $A, B, C \in \mathbf{H}_n$ such that $C = A + B$,

$$\lambda(A) = (a_1, \dots, a_n), \lambda(B) = (b_1, \dots, b_n), \lambda(C) = (c_1, \dots, c_n)$$

if and only if

$$\sum_{s=1}^n c_s = \sum_{s=1}^n (a_s + b_s)$$

and for any $(J_0, J_1, J_2) \in LR_m^n$ with $m < n$

$$\sum_{j \in J_0} c_j \leq \sum_{j \in J_1} a_j + \sum_{j \in J_2} b_j.$$

LR Sequences

Let $[n] = (1, \dots, n)$ and $J = (j_1, \dots, j_m)$ be an increasing subsequence of $[n]$, i.e., $1 \leq j_1 < \dots < j_m \leq n$. Define

$$\mu(J) = (j_m - m, \dots, j_1 - 1).$$

Suppose J_0, J_1, J_2 are increasing subsequences of $[n]$. Then $(J_0, J_1, J_2) \in LR_m^n$ if the Littlewood-Richardson coefficient $c_{\mu(J_1)\mu(J_2)}^{\mu(J_0)}$ of the three partitions $\mu(J_0), \mu(J_1)$, and $\mu(J_2)$ is positive, i.e., $\mu(J_0)$ can be generated from $\mu(J_1)$ and $\mu(J_2)$ according to the Littlewood-Richardson rules.

Littlewood-Richardson rules

Display $\mu(J_0) = (r_1, \dots, r_m)$, $\mu(J_1) = (s_1, \dots, s_m)$, and $\mu(J_2) = (t_1, \dots, t_m)$ as Young diagrams. Add $t_1 + \dots + t_m$ entries from $\{1, \dots, m\}$ to the rows of the Young diagram of $\mu(J_1)$ to generate the Young diagram of $\mu(J_0)$ so that:

1. The entries i occurs exactly t_i so many times.
2. The entries in each row is weakly increasing from left to right.
3. The entries in each column is strictly increasing from top to bottom.
4. For any p with $1 \leq p \leq \sum_{j=1}^m t_j$, define $p(i)$ to be the number of i in the first p assigned values counting from right to left and top to bottom, we have $p(i) \geq p(i+1)$.

Example Suppose

$$\mu(J_0) = (5, 4, 3, 1), \mu(J_1) = (3, 2, 1, 0), \mu(J_2) = (3, 2, 2, 0).$$

*	*	*	1	1	*	*	*	1	1	*	*	*	1	1
*	*	1	2		*	*	2	2		*	*	2	2	
*	2	3			*	1	3			*	3	3		
3					3					1				

			*	*	*	1	1
			*	*	1	2	
			*	3	3		
			2				

Example $((j_0), (j_1), (j_2)) \in LR_1^n$ if and only if $j_0 = j_1 + j_2 - 1$.

Example If $J_1 = (i_1, \dots, i_m)$ and $J_2 = (j_1, \dots, j_m)$ satisfy $i_m + j_m - m \leq n$, then $J_0 = (i_1 + j_1 - 1, \dots, i_m + j_m - m)$ is admissible.

Problem Generate the (J_0, J_1, J_2) sequences efficiently.

Remarks

By the results of Horn, Fulton, Knutson and Tao, etc.

- * One can focus on (J_0, J_1, J_2) sequences with LR coefficient equal to one.
- * One can use the Horn's consistent sequences (R, S, T) , which is defined recursively.

Definition of consistent sequences

Let $R = (r_1, \dots, r_m), S = (s_1, \dots, s_m), T = (t_1, \dots, t_m) \in [n]$.

1. For $m \geq 1$, $\sum_{\ell=1}^m r_\ell + m(m+1)/2 = \sum_{\ell=1}^m (s_\ell + t_\ell)$.

2. If $m > 1$, then for any consistent triple (U, V, W) :

$$U = (u_1, \dots, u_{m'}), V = (v_1, \dots, v_{m'}), W = (w_1, \dots, w_{m'})$$

with $m' \in \{1, \dots, m-1\}$, we have

$$\sum_{\ell=1}^{m'} r_{u_\ell} + m'(m'+1)/2 \geq \sum_{\ell=1}^{m'} (s_{v_\ell} + t_{w_\ell}).$$

Extension to the sum of r matrices

(Hermitian, real symmetric, quaternionic)

There are $A_1, \dots, A_r \in \mathbf{H}_n$ with $\lambda(A_s) = (a_1^{(s)}, \dots, a_n^{(s)})$ for $s = 1, \dots, r$, and $\lambda(\sum A_j) = (a_1^{(0)}, \dots, a_n^{(0)})$ if and only if

$$\sum_j a_j^{(0)} = \sum_s \sum_j a_j^{(s)}$$

and for any $(J_0, J_1, \dots, J_r) \in LR_m^n(r)$ with $m < n$

$$\sum_{j \in J_0} a_j^{(0)} \leq \sum_{s=1}^r \sum_{j \in J_s} a_j^{(s)}.$$

Note It is amazing that the set of inequalities are the same for complex Hermitian and real symmetric matrices.

For example, let \mathbf{S}_n be the set of $n \times n$ real matrices, and let $w = e^{2\pi i/n}$ and $D = \text{diag}(1, w, \dots, w^{n-1})$. Then for any $A \in \mathbf{S}_n$,

$$\text{diag}(A) = \frac{1}{n} \left(\sum_{j=1}^n D^j A (D^j)^* \right) = \frac{1}{n} (A_1 + \dots + A_n),$$

where $\lambda(A_j) = \lambda(A)$ for all j . So, the eigenvalues inequalities hold, and there are $B_1, \dots, B_n \in \mathbf{S}_n$ with $\lambda(B_j) = \lambda(A)$ such that

$$\text{diag}(A) = \frac{1}{n} (B_1 + \dots + B_n).$$

But even for $n = 3$, we don't know how to construct B_1, B_2, B_3 !

Question Can one derive an algorithm to do this?

How about a specific example?

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Find $B_1, B_2, B_3 \in \mathbf{S}_3$ with $\lambda(A) = \lambda(B_1) = \lambda(B_2) = \lambda(B_3)$ such that

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{3}(B_1 + B_2 + B_3).$$

2. Principal Submatrices of a Hermitian Matrix

Problem Study the relations between the eigenvalues of $A \in \mathbf{H}_n$ and those of its principal submatrices.

Well known special cases

- * [Schur-Horn] There is $A \in \mathbf{H}_n$ with the vector of diagonal entries (d_1, \dots, d_n) if and only if it is majorized by $\lambda(A)$.
- * [Fan-Pall] There is $A \in \mathbf{H}_n$ with an $m \times m$ principal submatrix $B \in \mathbf{H}_m$ such that $\lambda(A) = (a_1, \dots, a_n)$ and $\lambda(B) = (b_1, \dots, b_m)$ if and only if

$$a_j \geq b_j \geq a_{n-m+j} \quad \text{for } j = 1, \dots, m.$$

Theorem [LP, 2003] There is $A = \begin{pmatrix} A_1 & * \\ * & A_2 \end{pmatrix} \in \mathbf{H}_n$ such that $\lambda(A) = (a_1, \dots, a_n)$, $A_1 \in \mathbf{H}_k$ with $\lambda(A_1) = (a_1^{(1)}, \dots, a_k^{(1)})$, and $A_2 \in \mathbf{H}_{n-k}$ with $\lambda(A_2) = (a_1^{(2)}, \dots, a_{n-k}^{(2)})$ if and only if

$$\sum_j a_j = \sum_j a_j^{(1)} + \sum_j a_j^{(2)}$$

and for any $(J_0, J_1, J_2) \in LR_m^n$ with $m < n$

$$\sum_{j \in J_0} (a_j - a_n) \leq \sum_{j \in J_1} (a_j^{(1)} - a_n) + \sum_{j \in J_2} (a_j^{(2)} - a_n),$$

here $a_j^{(s)} = a_n$ whenever $j > n_s$.

Theorem [LP, 2003] Suppose $n_1 + \dots + n_r = n$. There exists $A = (A_{ij})_{1 \leq i, j \leq r} \in \mathbf{H}_n$ such that $\lambda(A) = (a_1, \dots, a_n)$, and $A_{jj} \in \mathbf{H}_{n_j}$ with $\lambda(A_{jj}) = (a_1^{(j)}, \dots, a_{n_j}^{(j)})$ for $j = 1, \dots, r$ if and only if

$$\sum_j a_j = \sum_s \sum_j a_j^{(s)}$$

and for any $(J_0, J_1, \dots, J_r) \in LR_m^n(r)$ with $m < n$

$$\sum_{j \in J_0} (a_j - a_n) \leq \sum_{s=1}^r \sum_{j \in J_s} (a_j^{(s)} - a_n),$$

here $a_j^{(s)} = a_n$ whenever $j > n_s$.

Remark One can reduce the list of inequalities.

* For each $s = 1, \dots, r$, only consider $J_s = (j_1^{(s)}, \dots, j_m^{(s)})$ such that either $j_m^{(s)} \leq n_s$ or the last p terms have the form: $n_s + 1, n_s + 2, \dots, n_s + p$.

* We did the case when $n_j \leq 2$.

Notation Suppose $A_{ii} \in \mathbf{H}_2$ has eigenvalues $a_1^{(i)} \geq a_2^{(i)}$ for $1 \leq i \leq m$, and $A_{ii} = [a_1^{(i)}] \in \mathbf{H}_1$ for $m + 1 \leq i \leq n - m$. Let (i_1, \dots, i_m) be a permutation of $(1, \dots, m)$ such that $a_2^{(i_1)} \geq \dots \geq a_2^{(i_m)}$. For any subset $R \subseteq \{1, \dots, m\}$ with $|R| = r$, let $b_1^R \geq \dots \geq b_{n-m-2r}^R$ be the eigenvalues of $\bigoplus_{i \notin R} A_i$.

Theorem There exists $A = (A_{ij}) \in \mathbf{H}_n$ with eigenvalues $c_1 \geq \dots \geq c_n$, such that $A_{ii} \in \mathbf{H}_2$ has eigenvalues $a_1^{(i)} \geq a_2^{(i)}$ for $1 \leq i \leq m$, and $A_{ii} = [a_1^{(i)}] \in \mathbf{H}_1$ for $m+1 \leq i \leq n-m$ if and only if

$$\sum_{i=1}^n c_i = \sum_{i=1}^{n-m} a_1^{(i)} + \sum_{i=1}^m a_2^{(i)}$$

and for any $(s, t) \in \{0, \dots, m\} \times \{0, \dots, n-2s\}$ with $0 < s+t < n$ and any s element subset $S \subseteq \{i_1, \dots, i_\ell\}$ with $\ell = \min\{m, s+t\}$, we have

$$\sum_{i=1}^t c_i + \sum_{i=t+2}^{s+t+1} c_i \geq \sum_{j \in S} a_2^{(j)} + \sum_{i=1}^t b_i^S.$$

3. Off-diagonal blocks

Problem Study when a matrix $X \in M_{k,n-k}$ can be the off-diagonal block of a matrix $C \in \mathbf{H}_n$ with prescribed eigenvalues.

Observation There is $C = \begin{pmatrix} * & X \\ X^* & * \end{pmatrix} \in \mathbf{H}_n$ with $\lambda(C) = (c_1, \dots, c_n)$ if and only if there are (for any) unitary matrices $U \in M_k$ and $V \in M_{n-k}$ the matrix

$$\tilde{C} = \begin{pmatrix} * & UXV \\ (UXV)^* & * \end{pmatrix} \in \mathbf{H}_n \text{ with } \lambda(\tilde{C}) = (c_1, \dots, c_n).$$

Denote by $s(X) = (s_1(X), \dots, s_k(X))$ the vector of singular values of $X \in M_{k,n-k}$ with entries arranged in descending order.

Theorem [LP, 2003] Let $c_1 \geq \cdots \geq c_n$ and $s_1 \geq \cdots \geq s_k \geq 0$ be given, where $k \leq n/2$. The following are equivalent.

- (a) There is $C = \begin{pmatrix} * & X \\ X^* & * \end{pmatrix} \in \mathbf{H}_n$ such that $X \in M_{k, n-k}$, $\lambda(C) = (c_1, \dots, c_n)$ and $s(X) = (s_1, \dots, s_k)$.
- (b) There exist $C_1, C_2, S \in \mathbf{H}_k$ such that $C_1 - C_2 \geq 2S$, where $\lambda(C_1) = (c_1, \dots, c_k)$, $\lambda(C_2) = (c_{n-k+1}, \dots, c_n)$, and $\lambda(S) = (s_1, \dots, s_k)$.
- (c) For each $(J_0, J_1, J_2) \in LR_m^k$ with $m \leq k$

$$2 \sum_{j \in J_0} s_j \leq \sum_{j \in J_1} c_j - \sum_{j \in J_2} c_{n-j+1}.$$

Some consequences

* Suppose $C = \begin{pmatrix} * & X \\ X^* & * \end{pmatrix} \in \mathbf{H}_n$ with $\lambda(C) = (c_1, \dots, c_n)$.

Then for any contractions $R \in M_k$ and $T \in M_{n-k}$, there

exists $\tilde{C} = \begin{pmatrix} * & RXT \\ (RXT)^* & * \end{pmatrix} \in \mathbf{H}_n$ with $\lambda(C) = \lambda(\tilde{C})$.

* In particular, taking $RXT = tX$ with $t \in [0, 1]$, we see that the set of matrices $X \in M_{k, n-k}$ which appear in the right top corners of matrices $C \in \mathbf{H}_n$ with eigenvalues c_1, \dots, c_n is star-shaped with the zero matrix as the star-center.

* [CLP, 2003] The set is convex if and only if (c_1, \dots, c_k) and (c_{n-k+1}, \dots, c_n) are arithmetic progressions with the same common difference.

4. Complex Symmetric Matrices

Problem

Study the singular values of the off-diagonal blocks of complex symmetric and general matrices.

Surprise

Sometimes, there are not much difference between complex symmetric or general matrices with real symmetric matrices!

Theorem [FFLP, '04] Let $\gamma_1 \geq \cdots \geq \gamma_n$ and $s_1 \geq \cdots \geq s_k \geq 0$ be given, where $k \leq n/2$. The following are equivalent.

- (a) There is a symmetric matrix $C = \begin{pmatrix} * & X \\ X^t & * \end{pmatrix} \in M_n$ with $X \in M_{k,n-k}$, $s(C) = (\gamma_1, \dots, \gamma_n)$ and $s(X) = (s_1, \dots, s_k)$.
- (b) There is $C = \begin{pmatrix} * & Y \\ Z & * \end{pmatrix} \in M_n$ with $s(C) = (\gamma_1, \dots, \gamma_n)$ such that $Y, Z \in M_{k,n-k}$ have a combined list of singular values: $s_1, s_1, s_2, s_2, \dots, s_k, s_k$.
- (c) There exists a real symmetric matrix $C = \begin{pmatrix} * & X \\ X^t & * \end{pmatrix}$ such that $X \in M_{k,n-k}$, $\lambda(C) = (\gamma_1, -\gamma_2, \gamma_3, \dots, (-1)^n \gamma_n)$ and $s(X) = (s_1, \dots, s_k)$.

(c) There exists a real symmetric matrix $C = \begin{pmatrix} * & X \\ X^t & * \end{pmatrix}$ such that $X \in M_{k,n-k}$, $\lambda(C) = (\gamma_1, -\gamma_2, \gamma_3 \dots, (-1)^n \gamma_n)$ and $s(X) = (s_1, \dots, s_k)$.

(d) There exist $C_1, C_2, S \in \mathbf{H}_k$ such that $C_1 + C_2 \geq 2S$, where $\lambda(C_1) = (\gamma_1, \gamma_3 \dots, \gamma_{2k-1})$, $\lambda(C_2) = (\gamma_2, \gamma_4, \dots, \gamma_{2k})$, and $\lambda(S) = (s_1, \dots, s_k)$.

(e) For each $(J_0, J_1, J_2) \in LR_m^k$ with $m \leq k$

$$2 \sum_{j \in J_0} s_j \leq \sum_{j \in J_1} \gamma_{2j-1} + \sum_{j \in J_2} \gamma_{2j}.$$

Some consequences

- * If $X \in M_{k, n-k}$ is the $(1, 2)$ block of a matrix in \mathbf{H}_n with eigenvalues values c_1, \dots, c_n , such that $|c_1| \geq \dots \geq |c_n|$, then X is the $(1, 2)$ block of a matrix in \mathbf{H}_n with eigenvalues $|c_1|, -|c_2|, |c_3|, -|c_4|, \dots$
- * If $A, B, C \in \mathbf{H}_n$ satisfies $C = A + B$ and the combined list of eigenvalues of A and B is $\gamma_1 \geq \dots \geq \gamma_{2n}$, then $C = \tilde{A} + \tilde{B}$ such that
$$\lambda(\tilde{A}) = (\gamma_1, \gamma_3, \dots, \gamma_{2n-1}) \text{ and } \lambda(\tilde{B}) = (\gamma_2, \gamma_4, \dots, \gamma_{2n}).$$
- * Same result work for $A_0 = A_1 + \dots + A_r$.
 - \tilde{A}_1 has eigenvalues $\gamma_1, \gamma_{r+1}, \gamma_{2r+1}, \dots$
 - \tilde{A}_2 has eigenvalues $\gamma_2, \gamma_{r+2}, \gamma_{2r+2}, \dots$
 - \tilde{A}_3 has eigenvalues $\gamma_3, \gamma_{r+3}, \gamma_{2r+3}, \dots$

5. Future research

- * Determine numerical algorithms to do the constructions.
- * Study the relations between the singular values of complementary blocks.
- * Study the relations between the singular values of C and those of S , or those of R and T , for

$$C = \begin{pmatrix} R & 0 \\ S & T \end{pmatrix}.$$

- * Study the implications of these results in the real world!

Thank you for your attention!