Numerical Range (Radius) Preserving Maps

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Introduction

$B(H)$: the set of all bounded linear operators on H.

$S(H)$: the set of all self-adjoint operators in $B(H)$.

M_n: the set of $n \times n$ matrices.

S_n: the set of $n \times n$ Hermitian matrices.

The numerical range of $A \in B(H)$ is the set

$$W(A) = \{(Ax, x) : x \in H,\ (x, x) = 1\},$$

and the numerical radius is the quantity

$$r(A) = \sup\{|\mu| : \mu \in W(A)\}.$$

Let $\mathbf{V} = B(H)$ or $S(H)$, let $F(A) = W(A)$ or $r(A)$.

Characterize mappings $\phi : \mathbf{V} \to \mathbf{V}$ which satisfy one of the following conditions.

1. ϕ is linear/additive/multiplicative and satisfies

$$F(\phi(A)) = F(A) \text{ for all } A \in \mathbf{V}.$$

2. $F(\phi(A) * \phi(B)) = F(A * B)$ for all $A, B \in \mathbf{V}$, where $A * B$ is any one of the following operations:

$$A + B,\ A - B,\ AB,\ ABA,\ (AB + BA)/2,\ \textit{etc.}$$

We are also interested in results on other domains and co-domains, and other types of numerical ranges and numerical radii.

Linear preservers

[Pellegrini, 1975], [Marcus and Filippenko, 1978], [Pierce and Watkins, 1979], [Li, 1987], [Chan, 1995], [Chan, 1998].

Theorem Let $\mathbf{V} = B(H), S(H)$, etc. A surjective linear preserver of $W(A)$ has the **standard form**

$$A \mapsto U^*AU \qquad \text{or} \qquad A \mapsto U^*A^tU,$$

A surjective linear preserver of $r(A)$ has the **standard form**

$$A \mapsto \mu U^*AU \qquad \text{or} \qquad A \mapsto \mu U^*A^tU.$$

No surjective assumption is needed in the finite dimensional case.

Additive preservers

[Lesnjak, 2001; Li and Šemrl, 2002; Bai and Hou,2003]

Theorem Let $\mathbf{V} = B(H)$ or $S(H)$. Surjective additive preservers of $W(A)$ have the standard form

$$A \mapsto U^* A U \quad \text{or} \quad A \mapsto U^* A^t U;$$

surjective additive preservers of $r(A)$ have one of the following forms:

$$A \mapsto \mu U^* A U, \qquad A \mapsto \mu U^* A^t U,$$

$$A \mapsto \mu U^* \overline{A} U, \quad \text{or} \quad A \mapsto \mu U^* A^* U.$$

In the finite dimensional case, the surjective assumption can be removed.

Sum and difference of operators

Let $\mathbf{V} = B(H)$ or $S(H)$, and let $F(A) = W(A)$ or $r(A)$.

1. A surjective map ϕ is real affine, i.e., $A \mapsto \phi(A) - \phi(0)$ is real linear, if it satisfies

$$F(\phi(A) - \phi(B)) = F(A - B) \text{ for all } A, B \in \mathbf{V}.$$

2. A surjective map ϕ is real linear if it satisfies

$$F(\phi(A) + \phi(B)) = F(A + B) \text{ for all } A, B \in \mathbf{V}.$$

In the finite dimensional case, the surjective assumption can be removed.

Products of operators

The following result will cover the multiplicative preservers.

[Di and Hou, 2004], [Cui and Hou, 2005], [Chan, Li, Sze, 2005]

Theorem Let $\mathbf{V} = B(H)$ and $S(H)$, $F(A) = W(A)$ or $r(A)$. Then a surjective map $\phi : \mathbf{V} \to \mathbf{V}$ satisfies

$$F(\phi(A)\phi(B)) = F(AB) \text{ for all } A, B \in \mathbf{V}$$

if and only if ϕ has the form

$$A \mapsto \mu_A U^* A U \qquad \textit{or} \qquad A \mapsto \mu_A U^* A^t U,$$

where U is unitary, $|\mu_A| = 1$, $\mu_A \in \{1, -1\}$ if $F(A) = W(A)$. In the finite dimensional case, the surjective assumption can be removed.

Jordan triple product [Di and Hou, 2004]

Theorem Let $\mathbf{V} = B(H)$ or $S(H)$. A surjective mapping $\phi : \mathbf{V} \to \mathbf{V}$ satisfies

$$W(\phi(A)\phi(B)\phi(A)) = W(ABA) \text{ for all } A, B \in \mathbf{V}$$

if and only if there is a unitary U and μ satisfying $\mu^3 = 1$ such that ϕ has the form

$$A \mapsto \mu U^* AU \quad \text{or} \quad A \mapsto \mu U^* A^t U.$$

Questions

(a) In the finite dimensional case, can we remove the surjective assumption?

(b) How about $r(A)$?

Suppose $W(ABA) = W(\phi(A)\phi(B)\phi(A))$.
Then $\phi(I)^3 = I$, $\phi(xx^*) = yy^*$?

Jordan product [Gau and Li, 2005]

Theorem Let $\mathbf{V} = B(H)$ or $S(H)$. A surjective mapping $\phi : \mathbf{V} \to \mathbf{V}$ satisfies

$$W(\phi(A)\phi(B) + \phi(B)\phi(A)) = W(AB + BA) \text{ for all } A, B \in \mathbf{V}$$

if and only if there is a unitary U such that ϕ has the form

$$A \mapsto \pm U^* AU \quad \text{or} \quad A \mapsto \pm U^* A^t U.$$

In the finite dimensional case, the surjective assumption can be removed.

Question How about $r(A)$?

Schur / Hadamard product [Li and E. Poon, 2005]

Theorem Let $\mathbf{V} = M_n$ or S_n. A mapping $\phi : \mathbf{V} \to \mathbf{V}$ satisfies

$$W(\phi(A) \circ \phi(B)) = W(A \circ B) \text{ for all } A, B \in \mathbf{V}$$

if and only if ϕ has the form

$$A \mapsto R \circ (D_A^* A D_A) \quad \text{or} \quad A \mapsto R \circ (D_A^* A^t D_A),$$

where $R \circ R = xx^*$ for a unit vector x, D_A is a diagonal unitary matrix depending on A.

Currently, working on $r(A)$.

Further research

1. Other types of numerical ranges (and numerical radii): k-numerical range, q-numerical range, c-numerical range, C-numerical range, norm numerical range, decomposable numerical range, etc.

2. Other types of domains/co-domains: SL_n, GL_n, block triangular matrices, block diagonal matrices, operator algebras, nest algebras, etc.

3. Consider $\phi : \mathbf{V} \to \mathbf{V}'$.

4. Consider other operations $A \bullet B$ such as
$$AB - BA,\ AB^*,\ AB^{-1}, \text{ etc.}$$

An incomplete list of researchers

Bai, Bebiano, Chan, Cheung, Cui, Di, Duffner, Filippenko, Guralnick, Hou, Lei, Man, Marcus, Mehta, Pellegrini, Poon, Providencia, Rodman, Šemrl, Sourour, Sze, B.S. Tam, T.Y. Tam, Tsing, Zaharia, . . .

Welcome to join the club!

Thank you for your attention!