Optimal Parameter in
Hermitian and Skew-Hermitian Splitting Method
for Certain Two-by-Two Block Matrices

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Joint Work with
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The HSS Iteration

Let $A = H + S \in \mathbb{C}^{n \times n}$ be a sparse matrix, where

$$H = (A + A^*)/2 \quad \text{and} \quad S = (A - A^*)/2,$$

so that $H$ is positive definite and $S \neq 0$. To solve the linear system

$$Ax = b,$$

consider the following HSS (Hemitian and Skew-Hermitian Spliting) iteration scheme proposed in [Bai,Golub,Ng, 2003].
The HSS Iteration

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consider the following HSS (Hemitian and Skew-Hermitian Spliting) iteration scheme proposed in [Bai, Golub, Ng, 2003].

Given an initial guess \( x^{(0)} \in \mathbb{C}^n \), compute \( x^{(k)} \) for \( k = 0, 1, 2, \ldots \) using the following iteration scheme until \( \{x^{(k)}\} \) satisfies the stopping criterion:

\[
\begin{cases}
(\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\
(\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b,
\end{cases}
\]

where \( \alpha \) is a given positive constant.
In matrix-vector form, we have

\[ x^{(k+1)} = M(\alpha)x^{(k)} + b(\alpha), \quad k = 0, 1, 2, \ldots, \]  

(1)

where

\[ b(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}b \]

and

\[ M(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S) \]  

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\[ x^{(k+1)} = \mathcal{M}(\alpha)x^{(k)} + b(\alpha), \quad k = 0, 1, 2, \ldots, \]

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Note that (1) may also result from the splitting

\[ A = B(\alpha) - C(\alpha) \]

of the coefficient matrix \( A \), with

\[ \begin{cases} 
B(\alpha) &= \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S), \\
C(\alpha) &= \frac{1}{2\alpha}(\alpha I - H)(\alpha I - S). 
\end{cases} \]
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have the same eigenvalues.

Furthermore, \(\widetilde{\mathcal{M}}(\alpha)\) has singular values
\[
\frac{1 - \lambda_j(H)}{1 + \lambda_j(H)}, \quad j = 1, \ldots, n.
\]
So, the spectral radius \(\rho(\mathcal{M}(\alpha)) = \rho(\widetilde{\mathcal{M}}(\alpha))\) is bounded above by
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\|\widetilde{\mathcal{M}}(\alpha)\| < 1.
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**Problem** How to find the optimal \( \alpha^* \) so that
\[
\rho(\mathcal{M}(\alpha^*)) \leq \rho(\mathcal{M}(\alpha)) \quad \text{for all } \alpha > 0.
\]
Let $\lambda_1$ and $\lambda_2$ be the maximum and minimum eigenvalues of $H$. If $\tilde{\alpha} = \sqrt{\lambda_1 \lambda_2}$, then
\[
\frac{\sqrt{\lambda_1} - \sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} = \|\tilde{M}(\tilde{\alpha})\| \leq \|\tilde{M}(\alpha)\|, \quad \alpha > 0.
\]

Thus,
\[
\rho(\mathcal{M}(\alpha^*)) \leq \rho(\mathcal{M}(\tilde{\alpha})) \leq \frac{\sqrt{\lambda_1} - \sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}}.
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In most situations,
\[
\rho(\mathcal{M}(\alpha^*)) \ll \frac{\sqrt{\lambda_1} - \sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}}.
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Problem Can we do better?
The two-by-two real case

**Theorem 1** Let \( A = H + S \in \mathbb{R}^{2 \times 2} \) be such that \( H \) is symmetric positive definite and \( S \) is skew-symmetric. Suppose \( H \) has eigenvalues \( \lambda_1 \geq \lambda_2 > 0 \) and \( \det(S) = q^2 \) with \( q \in \mathbb{R} \). Then the two eigenvalues of the iteration matrix \( M(\alpha) \) are

\[
\lambda_{\pm} = \frac{(\alpha^2 - \lambda_1 \lambda_2)(\alpha^2 - q^2) \pm \sqrt{\Delta(\alpha)}}{(\alpha + \lambda_1)(\alpha + \lambda_2)(\alpha^2 + q^2)},
\]

where

\[
\Delta(\alpha) = (\alpha^2 - \lambda_1 \lambda_2)^2(\alpha^2 - q^2)^2 - (\alpha^2 - \lambda_1^2)(\alpha^2 - \lambda_2^2)(\alpha^2 + q^2)^2.
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\]

As a result,

\[
\rho(M(\alpha)) = \begin{cases} 
\left| (\alpha^2 - \lambda_1 \lambda_2)(\alpha^2 - q^2) \right| + \sqrt{\Delta(\alpha)} & \text{if } \Delta(\alpha) \geq 0; \\
\sqrt{(\alpha - \lambda_1)(\alpha - \lambda_2)} & \text{if } \Delta(\alpha) < 0.
\end{cases}
\]
Proof. Apply an orthogonal similarity, and assume that

\[ H = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix} \quad \text{with} \ q \in \mathbb{R}. \]

Then \((\alpha I + H)^{-1}(\alpha I - H)(\alpha I + S)^{-1}(\alpha I - S)\) equals

\[
\frac{1}{\alpha^2 + q^2} \cdot \begin{bmatrix}
\frac{(\alpha^2 - q^2)(\alpha - \lambda_1)}{\alpha + \lambda_1} & -\frac{2q\alpha(\alpha - \lambda_1)}{\alpha + \lambda_1} \\
\frac{2q\alpha(\alpha - \lambda_2)}{\alpha + \lambda_2} & \frac{(\alpha^2 - q^2)(\alpha - \lambda_2)}{\alpha + \lambda_2}
\end{bmatrix}.
\]

The formula for \(\lambda_{\pm}\) and the assertion on \(\rho(M(\alpha))\) follow. \(\square\)
One may want to use the formula of $\rho(\mathcal{M}(\alpha))$ in Theorem 1 to determine the optimal choice of $\alpha$. It turns out that the analysis is very complicated and not productive. The main difficulty is the expression

$$\sqrt{\Delta(\alpha)} = \sqrt{(\alpha^2 - \lambda_1 \lambda_2)^2(\alpha^2 - q^2)^2 - (\alpha^2 - \lambda_1^2)(\alpha^2 - \lambda_2^2)(\alpha^2 + q^2)^2}$$

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Here, we use a different approach that allows us to avoid the complicated expression (3). The key idea is:
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If $X$ has eigenvalues $\gamma_1, \gamma_2$ then $\rho(X) = \max\{\vert\gamma_1\vert, \vert\gamma_2\vert\}$ so that

$$\rho(X) \geq \vert(\gamma_1 + \gamma_2)/2\vert = \vert(\text{tr } X)/2\vert$$

and

$$\rho(X)^2 \geq \vert\gamma_1 \gamma_2\vert = \vert\det(X)\vert.$$
For notational simplicity, write

\[ \rho(\alpha) = \rho(M(\alpha)), \]

\[ \tau(\alpha) = \left\{ \frac{\text{trace}(M(\alpha))}{2} \right\}^2 = \left\{ \frac{(\alpha^2 - q^2)(\alpha^2 - \lambda_1 \lambda_2)}{(\alpha^2 + q^2)(\alpha + \lambda_1)(\alpha + \lambda_2)} \right\}^2, \]

\[ \delta(\alpha) = |\det(M(\alpha))| = \left| \frac{(\alpha - \lambda_1)(\alpha - \lambda_2)}{(\alpha + \lambda_1)(\alpha + \lambda_2)} \right|, \]

and

\[ \omega(\alpha) = \max\{\tau(\alpha), \delta(\alpha)\}. \]

Then

\[ \rho(\alpha)^2 \geq \omega(\alpha). \]
Note that
\[ 1 = \tau(0) = \lim_{\alpha \to +\infty} \tau(\alpha) \quad \text{and} \quad 1 = \delta(0) = \lim_{\alpha \to +\infty} \delta(\alpha). \]
Thus,
\[ \lim_{\alpha \to +\infty} \omega(\alpha) = \omega(0) = 1 > \omega(\xi) \quad \text{for all} \ \xi > 0. \]
Since \( \omega(\alpha) \) is continuous and nonnegative, there exists \( \alpha^* > 0 \) such that
\[ \omega(\alpha^*) = \min\{\omega(\alpha) : \alpha > 0\}. \]
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As a result, the eigenvalues of \( M(\alpha^*) \) have the same modulus, and thus
\[ \rho(\alpha)^2 \geq \omega(\alpha) \geq \omega(\alpha^*) = \rho(\alpha^*)^2, \quad \text{for all} \ \alpha > 0. \]
**Theorem 2** Let the assumptions of Theorem 1 be satisfied and define the functions \( \tau \) and \( \delta \) as above. Then the optimal \( \alpha^* > 0 \) satisfying

\[
\rho(M(\alpha^*)) = \min\{\rho(M(\alpha)) : \alpha > 0\}
\]

lies in the finite set

\[
S = \{\alpha > 0 : \tau(\alpha) = \delta(\alpha)\},
\]

which consists of numbers \( \alpha > 0 \) satisfying

\[
(\alpha^2 + q^2)^2(\alpha^2 - \lambda_1^2)(\alpha^2 - \lambda_2^2) = (\alpha^2 - q^2)^2(\alpha^2 - \lambda_1 \lambda_2)^2 \quad (5)
\]
or

\[
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Proof. Two pages of detailed analysis. \( \square \)
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**Proof.** Two pages of detailed analysis. \( \square \)

**Remark** Use the substitution \( \beta = \alpha^2 \) in (5) and (6) to get degree two/three polynomial equations!
Applications to two-by-two block matrices

**Theorem 3** Suppose \( A = H + S \in \mathbb{C}^{n \times n} \) such that
\[
H = \frac{1}{2}(A + A^*) = \begin{bmatrix} \lambda_1 I_r & 0 \\ 0 & \lambda_2 I_s \end{bmatrix} \quad \text{and} \quad S = \frac{1}{2}(A - A^*) = \begin{bmatrix} 0 & E \\ -E^* & 0 \end{bmatrix},
\]
where \( \lambda_1 > \lambda_2 > 0 \), and the nonzero matrix \( E \in \mathbb{C}^{r \times s} \) has nonzero singular values \( q_1 \geq q_2 \geq \cdots \geq q_k \). Then the spectral radius of the iteration matrix
\[
\mathcal{M}(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)
\]
attains the minimum at \( \alpha^* \), which is \( \sqrt{\lambda_1 \lambda_2}, \sqrt{q_1 q_k} \), or a root of one of the following equations:
\[
(\alpha^2 + q_j^2)^2(\alpha^2 - \lambda_1^2)(\alpha^2 - \lambda_2^2) = (\alpha^2 - q_j^2)^2(\alpha^2 - \lambda_1 \lambda_2)^2
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where \( j = 1, k \).
Estimation of optimal parameters for $n$-by-$n$ matrices

In general, for a nonsymmetric and positive definite system of linear equations $Ax = b$, the eigenvalues of its coefficient matrix $A$ lies in

$$D = \{x + iy : \lambda_1 \geq x \geq \lambda_2, -q \leq y \leq q\},$$

where $\lambda_1$ and $\lambda_2$ are the largest and the smallest eigenvalues of $H$, and $q$ is the largest module of the eigenvalues of the skew-Hermitian part $S$, of the coefficient matrix $A$. 
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A reduced (simpler and lower-dimensional) matrix $A_R$ whose eigenvalues possess the same contour as the domain $D$ is used to approximate the matrix $A$. For instance, a simple choice of the reduced matrix is given by

$$A_R = \begin{bmatrix} \lambda_1 & q \\ -q & \lambda_2 \end{bmatrix} \quad \text{with} \quad q = \|S\| \quad \text{or} \quad q = \rho(H^{-1}S)\sqrt{\lambda_1\lambda_2}.$$ 

We then use our results to estimate the optimal parameter $\alpha^*$ of the HSS iteration method as follows.
Let $A \in \mathbb{R}^{n \times n}$ be a positive definite matrix, and $H, S \in \mathbb{R}^{n \times n}$ be its symmetric and skew-symmetric parts, respectively. Let $\lambda_1$ and $\lambda_2$ be the largest and smallest eigenvalues of $H$. Suppose $q = \|S\| \quad \text{or} \quad q = \rho(H^{-1}S)\sqrt{\lambda_1 \lambda_2}$.

Then one can use the positive roots of the equation

$$(\alpha^2 + q^2)^2(\alpha^2 - \lambda_1^2)(\alpha^2 - \lambda_2^2) = (\alpha^2 - q^2)^2(\alpha^2 - \lambda_1 \lambda_2)^2$$

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Numerical examples were given to illustrate that the estimations are useful.
Further research

Determine the optimal parameters for other classes of matrices $A$. 
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Thank you for your attention!!!