Research with Undergraduates

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Sources of undergraduate research students

* Summer REU program supported by NSF, students from other universities, 8 weeks.
* William and Mary summer research program, William and Mary students, 6 to 8 weeks.
* Research during the academic year supported by NSF, William and Mary students, one to two semesters.
* Honors projects of William and Mary mathematics majors, two semesters.

How to choose problems?

I mentioned some sample problems, or students suggested problems they were interested in.

Preserver problems

$\mathbb{F}$: the set of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers,

$\mathbb{F}^n$: the set of $n \times 1$ vectors with entries in $\mathbb{F}$,

$M_{m,n} = M_{m,n}(\mathbb{F})$: the set of $m \times n$ matrices over $\mathbb{F}$,

Use the notation $M_n = M_n(\mathbb{F})$ if $m = n$.

General problem Characterize transformations on matrices with special properties.

Hint Always think about real 2×2 matrices.

Determinant preservers

Let $M, N \in M_n$ with $\det(MN) = 1$. Define $\phi : M_n \to M_n$ by

$$\phi(A) = MAN \quad \text{for all } A \in M_n, \text{ or}$$

$$\phi(A) = MA^tN \quad \text{for all } A \in M_n.$$

Then

$$\det(A) = \det(\phi(A)) \quad \text{for all } A \in M_n. \tag{†}$$

Problem Find all linear transformations ϕ satisfying (†).

By the (surprising) result in [Frobenius, 1897], the converse is also true.

Eigenvalue preservers

Let $\mathrm{Eig}\,(A)$ be the set of eigenvalues of $A \in M_n$.

Let $S \in M_n$ be an invertible matrix. Define $\phi : M_n \to M_n$ by

$$\phi(A) = SAS^{-1} \quad \text{for all } A \in M_n, \text{ or}$$

$$\phi(A) = SA^tS^{-1} \quad \text{for all } A \in M_n.$$

Then

$$\mathrm{Eig}\,(\phi(A)) = \mathrm{Eig}\,(A) \quad \text{for all } A \in M_n.$$

By the result in [Marcus & Purves, 1959], the converse is true.

Singular matrices preservers

[Dieudonné, 1949] Let R be the set of singular matrices in M_n. A bijective linear transformation $\phi : M_n \to M_n$ satisfies

$$\phi(R) \subseteq R$$

if and only if there are invertible matrices $M, N \in M_n$ such that

$$\phi(A) = MAN \quad \text{for all } A \in M_n, \text{ or}$$

$$\phi(A) = MA^tN \quad \text{for all } A \in M_n.$$

Invertible matrices preservers

[Marcus and Purves, 1959] Let R be the set of invertible matrices in M_n. A linear transformation $\phi : M_n \to M_n$ satisfies

$$\phi(R) \subseteq R$$

if and only if there are $M, N \in R$ such that

$$\phi(A) = MAN \quad \text{for all } A \in M_n, \text{ or}$$

$$\phi(A) = MA^tN \quad \text{for all } A \in M_n.$$

Rank preservers

[Marcus, Moyls, Djokovic, Lim, Beasley, Laffey, 1959-1994]
Let $1 \le k \le n$ and R_k be the set of rank k matrices in M_n. A linear transformation $\phi : M_n \to M_n$ satisfies

$$\phi(R_k) \subseteq R_k$$

if and only if there are invertible matrices M and N such that

$$\phi(A) = MAN \quad \text{for all } A \in M_n, \text{ or}$$

$$\phi(A) = MA^tN \quad \text{for all } A \in M_n.$$

Remark In the complex case, bijectivity assumption is not needed.

Adjacency preservers

[Hua, 1951] A bijective map $\phi : M_n \to M_n$ satisfies the condition that

$$\text{rank}\,(\phi(A) - \phi(B)) = 1 \text{ if and only if } \text{rank}\,(A - B) = 1$$

if and only if there are invertible matrices $M, N, S \in M_n$ with $\det(MN) \neq 0$ such that

$$\phi(A) = MAN + S \quad \text{ for all } A \in M_n, \text{ or}$$

$$\phi(A) = MA^tN + S \quad \text{ for all } A \in M_n.$$

There are many other results and interesting (difficult) open problems. Can you make up some problems?

In the following, I discuss some of my recent work with colleagues and students on this topic.

Preservers of permutation matrices

An $n \times n$ real matrix P is a permutation matrix if each row and each column of it has exactly one nonzero entry equal to one. Let S_n be the set of $n \times n$ permutation matrices.

Example I_3, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

[Li,Tam,Tsing, 2002] Let V_n be the linear space generated by S_n. A linear transformation $\phi : V_n \to V_n$ satisfies $\phi(S_n) = S_n$ if and only if there are $P, Q \in S_n$ such that

$$\phi(A) = PAQ \quad \text{for all } A \in M_n, \text{ or}$$

$$\phi(A) = PA^tQ \quad \text{for all } A \in M_n.$$

Question Let A_n be the set of all permutation matrices with determinant one. Characterize $\phi : V_n \to V_n$ such that $\phi(A_n) = A_n$.

If $n = 2$, there is only one matrix in A_2.

If $n = 3$, there are 3 matrices in S_3. Just send the 3 matrices to the three matrices.

If $n = 4$, there are 12 matrices in $S_4 = R_1 \cup R_2 \cup R_3$.

Each R_j has 4 elements. Send R_1, R_2, R_3 to R_i, R_j, R_k.

If $n \geq 5$, then ...

[Chiang and Li, 2002] Let V_n be the linear space generated by A_n. A linear transformation $\phi : V_n \to V_n$ satisfies $\phi(A_n) = A_n$ if and only if there are $P, Q \in S_n$ with $\det(PQ) = 1$ such that

$$\phi(A) = PAQ \quad \text{for all } A \in M_n, \text{ or}$$

$$\phi(A) = PA^tQ \quad \text{for all } A \in M_n.$$

Remark The result is related to group theory.

Chiang and I have papers on

* preservers of symmetric permutation and symmetric doubly stochastic matrices, etc.
* preservers of essentially doubly stochastic matrices.

Another direction

Note that $P, Q \in S_n$ ensures that $PQ \in S_n$, and $P, Q \in A_n$ ensures that $PQ \in A_n$.

Problem Let $G = S_n$ or A_n. Characterize bijective $\phi : G \to G$ such that $\phi(XY) = \phi(X)\phi(Y)$ for all $X, Y \in G$.

Suppose $n \neq 6$. There are $P \in S_n$ such that

$$\phi(A) = PAP^t \quad \text{for all } A \in G.$$

Results in this direction

[Alwill, Li, Maher, Sze, 2003/04] characterized multiplicative transformations satisfying $\phi(G) = G$, where G is the set of

- doubly stochastic, row stochastic, column stochastic matrices;
- essentially row, column, doubly stochastic matrices;
- monomial, nonnegative monomial matrices;
- nonnegative, positive matrices, etc.

Further research

Study linear/multiplicative/general transformations $\phi : \mathcal{M} \to \mathcal{M}'$ with special properties, say, preserving determinant, invertible matrices, spectra, etc.

Other topics

Distance preserving maps (isometries) on $\mathbb{F}^n$ or $M_{m,n}$.

Problems involving zero-one matrices.

Convex matrix sets.

Numerical ranges.

Finite reflection groups, central groupoids.

Game theory, coding theory.

Visit my webpage – http://www.math.wm.edu/~ckli

and send me your questions – ckli@math.wm.edu

THANK YOU FOR YOUR ATTENTION!