Eigenvalues and Singular Values of Perturbed Matrices

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Introduction

H_n: the set of $n \times n$ Hermitian matrices.

Denote the eigenvalues of $A \in H_n$ by

$$\lambda_1(A) \geq \cdots \geq \lambda_n(A).$$

Theorem [Weyl] Suppose $\tilde{A} = A + E$ with $A, E \in H_n$. Then for any $1 \leq j \leq n$, we have

$$\lambda_n(E) \leq \lambda_j(\tilde{A}) - \lambda_j(A) \leq \lambda_1(E).$$

Consider the Hermitian matrices:

$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \quad \text{have eigenvalues} \quad \lambda_1 \geq \cdots \geq \lambda_k,$$

$$\widetilde{A} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \quad \text{have eigenvalues} \quad \widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_k.$$

Suppose

$\|E\|$ is the spectral norm of the matrix E,

$\mathrm{Sp}\,(X)$ is the spectrum of the square matrix X,

$$\eta = \min\{|\mu_1 - \mu_2| : \mu_1 \in \mathrm{Sp}\,(H_1), \mu_2 \in \mathrm{Sp}\,(H_2)\}$$
$$= \text{the spectral gap between the spectra of } H_1 \text{ and } H_2.$$

There are two kinds of perturbation bounds on $|\lambda_i - \widetilde{\lambda}_i|$:

1. $|\lambda_i - \widetilde{\lambda}_i| \leq \|E\|$, and
2. $|\lambda_i - \widetilde{\lambda}_i| \leq \|E\|^2/\eta$
   if there is gap η between $\mathrm{Sp}\,(H_1)$ and $Sp(H_2)$.

The first bound overestimates the changes when $\|E\| \ll \eta$. The second bound blows up when η is small.

Our **goal** is to show that for $i = 1, \ldots, k$,

$$|\lambda_i - \widetilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}},$$

and obtain similar bounds for singular values of matrices under block perturbations.

An example

Consider the 2×2 real matrix $A = \begin{pmatrix} \alpha & \epsilon \\ \epsilon & \beta \end{pmatrix}$ with $\alpha > \beta$.

The eigenvalues of A are the roots of

$$\lambda^2 - (\alpha + \beta)\lambda + \alpha\beta - \epsilon^2 = 0.$$

Thus,

$$\lambda_\pm = \tfrac{\alpha+\beta\pm\sqrt{(\alpha+\beta)^2-4(\alpha\beta-\epsilon^2)}}{2} = \tfrac{\alpha+\beta\pm\sqrt{(\alpha-\beta)^2+4\epsilon^2}}{2}.$$

Hence

$$\begin{aligned}
0 < \lambda_+ - \alpha &= \beta - \lambda_- \\
&= \frac{-(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + 4\epsilon^2}}{2} \\
&= \frac{2\epsilon^2}{(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + 4\epsilon^2}} \\
&= \frac{2\epsilon^2}{\eta + \sqrt{\eta^2 + 4\epsilon^2}} \\
&\le \min\{\epsilon, \epsilon^2/\eta\},
\end{aligned}$$

which converges to ϵ as $\alpha \to \beta^+$.

Theorem *Consider the Hermitian matrices*

$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \quad \textit{have eigenvalues} \quad \lambda_1 \geq \cdots \geq \lambda_k, \textit{ and}$$

$$\widetilde{A} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \quad \textit{have eigenvalues} \quad \widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_k.$$

where $H_1 \in M_m,\ H_2 \in M_n$. *For* $i = 1, \ldots, m+n$,

$$|\lambda_i - \widetilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}},$$

where

$$\eta_i = \begin{cases} \min\{|\widetilde{\lambda}_i - \mu| : \mu \in \mathrm{Sp}\,(H_1)\} & \textit{if } \widetilde{\lambda}_i \in \mathrm{Sp}\,(H_2), \\ \min\{|\widetilde{\lambda}_i - \mu| : \mu \in \mathrm{Sp}\,(H_2)\} & \textit{if } \widetilde{\lambda}_i \in \mathrm{Sp}\,(H_1). \end{cases}$$

Outline of proof.

Step 1. Reclacing A by $(U_1 \oplus U_2)^* A (U_1 \oplus U_2)$ by suitable $U_1 \oplus U_2$, we may assume that H_1 and H_2 are diagonal matrices with diagonal entries arranged in descending order.

Step 2. We may perturb the diagonal entries and assume that they are distint. The general case will follow by continuity.

Step 3. We prove the result by induction on $m+n$.
The result is true if $m+n=2$ by the example.
Assume that $m+n>2$, and the result is true for Hermitian matrices of size at most $m+n-1$.

Step 3.a Suppose $A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix}$. Then $\lambda_1 I - A$ has zero as the smallest eigenvalue. Assume $\tilde{\lambda}_1$ occurs in H_1. By Schur complement,

$$\lambda_1 I_m - H_1 - E^*(\lambda_1 I_n - H_2)^{-1} E$$

has zero as the smallest eigenvalue. So,

$$\delta_1 = \lambda_1 - \tilde{\lambda}_1 = \lambda_m(\lambda_1 I_m - H_1) \leq \lambda_1(E^*(\lambda_1 I_n - H_2)^{-1} E).$$

Hence

$$\delta_1 \leq \lambda_1(E^*(\lambda_1 I_n - H_2)^{-1} E) \leq \|E\|^2/(\delta_1 + \eta_1).$$

and

$$\delta_1 \leq \frac{2\|E\|}{\eta_1 + \{\eta_1^2 + 4\|E\|^2\}^{1/2}}.$$

Step 3.b Suppose $1 < i < m+n$.
The result trivially holds if $\lambda_i = \widetilde{\lambda}_i$. So, assume that $\widetilde{\lambda}_i > \lambda_i$. Otherwise, replace $(A, \widetilde{A}, i)$ by $(-A, -\widetilde{A}, m+n-i+1)$.
Delete the row and column of A that contain the diagonal entry $\widetilde{\lambda}_n$ to get

$$\widehat{A} = \begin{pmatrix} \widehat{H}_1 & \widehat{E}^* \\ \widehat{E} & \widehat{H}_2 \end{pmatrix}$$

with eigenvalues $\nu_1 \geq \cdots \geq \nu_{m+n-1}$. By the interlacing inequalities $\lambda_i \geq \nu_i$ and hence

$$\widetilde{\lambda}_i - \lambda_i \leq \widetilde{\lambda}_i - \nu_i. \tag{1}$$

Now, $\widetilde{\lambda}_i$ is the ith largest diagonal entries in $\widehat{A}$ and

$$\widehat{\eta}_i \geq \eta_i.$$

Also, note that $\|\widehat{E}\| \leq \|E\|$. Thus,

$$\begin{aligned}
|\lambda_i - \widetilde{\lambda}_i| &= \widetilde{\lambda}_i - \lambda_i && \text{because } \widetilde{\lambda}_i > \lambda_i \\
&\leq \widetilde{\lambda}_i - \nu_i && \text{by (1)} \\
&\leq \frac{2\|\widehat{E}\|^2}{\widehat{\eta}_i + \sqrt{\widehat{\eta}_i^2 + 4\|\widehat{E}\|^2}} && \text{by induction assumption} \\
&\leq \frac{2\|\widehat{E}\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|\widehat{E}\|^2}} && \text{because } \widehat{\eta}_i \geq \eta_i \\
&= \frac{1}{2}\sqrt{\eta_i^2 + 4\|\widehat{E}\|^2} - \eta_i && \\
&\leq \frac{1}{2}\sqrt{\eta_i^2 + 4\|E\|^2} - \eta_i && \text{because } \|\widehat{E}\| \leq \|E\| \\
&= \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}}. && \blacksquare
\end{aligned}$$

Singular values

Denote the sequence of singular values of a complex $p \times q$ matrix X by $\sigma(X) = (\sigma_1(X), \ldots, \sigma_k(X))$, where $k = \max\{p, q\}$.

Theorem Let

$B = \begin{pmatrix} G_1 & E_1 \\ E_2 & G_2 \end{pmatrix}$ and $\widetilde{B} = \begin{pmatrix} G_1 & O \\ O & G_2 \end{pmatrix}$ have singular values

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ and $\widetilde{\sigma}_1 \geq \widetilde{\sigma}_2 \geq \cdots \geq \widetilde{\sigma}_r$, respectively,

where $G_1 \in M_{m,k}$, $G_2 \in M_{n,\ell}$ are non-trivial,
and $r = \max\{m + n, k + \ell\}$.

Define $\epsilon = \max\{\|E_1\|, \|E_2\|\}$, and

$$\eta_i = \begin{cases} \min\limits_{\mu \in \sigma(G_2)} |\widetilde{\sigma}_i - \mu| & \text{if } \widetilde{\sigma}_i \in \sigma(G_1), \\ \min\limits_{\mu \in \sigma(G_1)} |\widetilde{\sigma}_i - \mu| & \text{if } \widetilde{\sigma}_i \in \sigma(G_2), \end{cases}$$

and $\qquad \eta = \min_{1 \leq i \leq m+n} \eta_i.$

Then for $i = 1, 2, \cdots, \min\{m+n, k+\ell\}$, we have

$$|\sigma_i - \widetilde{\sigma}_i| \leq \frac{2\epsilon^2}{\eta_i + \sqrt{\eta_i^2 + 4\epsilon^2}} \leq \frac{2\epsilon^2}{\eta + \sqrt{\eta^2 + 4\epsilon^2}},$$

and $\sigma_i = \widetilde{\sigma}_i = 0$ for $i > \min\{m+n, k+\ell\}$.

Theorem Suppose $B = (\, G \; E \,)$ and $\tilde{B} = (\, G \; O \,)$ are $p \times q$ matrices with singular values

$$\sigma_1 \geq \cdots \geq \sigma_{\max\{p,q\}} \quad \text{and} \quad \widetilde{\sigma}_1 \geq \ldots \geq \widetilde{\sigma}_{\max\{p,q\}},$$

respectively. Then for $i = 1, \ldots, \min\{p, q\}$,

$$|\sigma_i - \widetilde{\sigma}_i| \leq \frac{2\|E\|^2}{2\widetilde{\sigma}_i + \sqrt{\widetilde{\sigma}_i^2 + 4\|E\|^2}}.$$