

Isometries for Ky Fan k -norms

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This is based on joint work with many colleagues:

Chan, Cheung, Djokovic, Guralnick, Johnson, Laffey,
Poon, Šemrl, Sourour, Sze, Tsing, Tu,

Singular values and Ky Fan k -norms

$M_{m,n}$ (M_n): $m \times n$ ($n \times n$) matrices over $\mathbb{F} = \mathbb{C}$ or \mathbb{R} ,

$\mathcal{U}_k = \{U \in M_k : U^*U = I_k\}$, here $A^* = A^t$ if $\mathbb{F} = \mathbb{R}$.

For our purpose, we may always assume that $m \leq n$.

The *singular values* of $A \in M_{m,n}$ are the nonnegative square roots of the eigenvalues of AA^* :

$$s_1(A) \geq \cdots \geq s_m(A).$$

Let $k \leq m$. The *Ky Fan k -norm* of $A \in M_{m,n}$ is

$$F_k(A) = \sum_{j=1}^k s_j(A).$$

Special cases

When $k = 1$, we get the *operator norm* (*spectral norm*)

$$\|A\|_{op} = \max\{\ell_2(Ax) : x \in \mathbb{C}^n, \ell_2(x) = 1\} = s_1(A).$$

When $n = m = k$, we get the *trace norm*

$$\|A\|_{tr} = \sum_{j=1}^n s_j(A).$$

UI norms and Ky Fan dominance theorem

A norm $\|\cdot\|$ on $M_{m,n}$ is *unitarily invariant* if

$$\|UAV\| = \|A\|$$

for any $A \in M_{m,n}$, $U \in \mathcal{U}_m$, and $V \in \mathcal{U}_n$.

Theorem [Ky Fan, 1951] Let $A, B \in M_{m,n}$. Then

$$\|A\| \leq \|B\| \text{ for all unitarily invariant norm } \|\cdot\|$$

if and only if

$$F_k(A) \leq F_k(B) \text{ for } k = 1, \dots, m.$$

Isometries

A linear map $\phi : M_{m,n} \rightarrow M_{m,n}$ satisfying

$$\|\phi(A)\| = \|A\| \quad \text{for all } A \in M_{m,n}$$

is an *isometry* for $\|\cdot\|$.

Note that ϕ is an isometry for $\|\cdot\|$ if and only if $\phi(\mathcal{B}) = \mathcal{B}$, where

$$\mathcal{B} = \{X \in M_{m,n} : \|X\| \leq 1\}.$$

The set of isometries is a group, which is called the *isometry group* of $\|\cdot\|$.

Why study isometries

- * Isometries are the **natural** mappings from a normed vector space to themselves.
- * Isometries can be viewed as the symmetries of the normed vector space. Characterizing the isometries will enhance our understanding of the geometry of the space, which is **useful** in the study of approximation problems, optimization problems, etc.
- * The study **stimulate interactions** among different areas: (differential) geometry, group theory, (norm) numerical range.

Isometries for the operator norm

Theorem [Kadison 1951; Marcus 1959] Suppose $\mathbb{F} = \mathbb{C}$. Let $\phi : M_n \rightarrow M_n$ be linear. TFAE.

(a) $\|\phi(A)\|_{op} = \|A\|_{op}$ for all $A \in M_n$.

(b) There exist unitary $U, V \in M_n$ such that ϕ has the form

$$A \mapsto UAV \quad \text{or} \quad A \mapsto UA^tV.$$

(c) $\phi(\mathcal{U}_n) = \mathcal{U}_n$ or $\phi(\mathcal{U}_n) \subseteq \mathcal{U}_n$.

Remark If $\phi(I) = I$ and ϕ is an isometry for the operator norm on $M_n(\mathbb{C})$ then ϕ is a C^* -isomorphism, i.e., ϕ has the form

$$A \mapsto V^*AV \quad \text{or} \quad A \mapsto V^*A^tV.$$

Remarks

- * Kadison reduced the problem to state preserving maps, i.e.,

$$\phi(\mathcal{S}) = \mathcal{S} = \{xx^* : x \in \mathbb{C}^n, x^*x = 1\}.$$

- * The set \mathcal{U}_n is the set of extreme points of the unit ball

$$\mathcal{B}_{op} = \{A \in M_n : \|A\|_{op} \leq 1\}.$$

An isometry ϕ for $\|\cdot\|_{op}$ will satisfy $\phi(\mathcal{B}_{op}) = \mathcal{B}_{op}$ and $\phi(\mathcal{U}_n) = \mathcal{U}_n$.

- * Using the result of [Wei 1972] on linear preservers of real orthogonal group, one gets the result for the real case.

Isometries for the trace norm

Theorem [Marcus 1959, Russo 1976] Suppose $\mathbb{F} = \mathbb{C}$. Let $\phi : M_n \rightarrow M_n$ be linear. TFAE.

(a) $\|\phi(A)\|_{tr} = \|A\|_{tr}$ for all $A \in M_n$.

(b) There exist unitary $U, V \in M_n$ such that ϕ has the form

$$A \mapsto UAV \quad \text{or} \quad A \mapsto UA^tV.$$

(c) $\phi(\mathcal{R}) = \mathcal{R}$ or $\phi(\mathcal{R}) \subseteq \mathcal{R}$, where

$$\mathcal{R} = \{xy^* : x, y \in \mathbb{C}^n, x^*x = y^*y = 1\}.$$

Remarks

- * The set \mathcal{R} is of extreme points of the unit ball

$$\mathcal{B}_{tr} = \{A \in M_n : \|A\|_{tr} \leq 1\}.$$

An isometry ϕ for $\|\cdot\|_{tr}$ will satisfy $\phi(\mathcal{B}_{tr}) = \mathcal{B}_{tr}$ and $\phi(\mathcal{R}) = \mathcal{R}$.

- * Using the result on linear preservers of real rank one matrices, one gets the result for the real case.

A Duality Technique

On $M_{m,n}$, consider the standard inner product

$$\langle X, Y \rangle = \text{tr}(AB^*).$$

The dual norm of a norm $\|\cdot\|$ on $M_{m,n}$ is the norm

$$\|A\|^* = \max\{|\langle A, B \rangle| : \|B\| \leq 1\}.$$

The *dual transformation* of a linear map $\phi : M_{m,n} \rightarrow M_{m,n}$ is the unique linear map $\phi^* : M_{m,n} \rightarrow M_{m,n}$ such that

$$\langle \phi(A), B \rangle = \langle A, \phi^*(B) \rangle \text{ for all } A, B \in M_{m,n}.$$

Remark A linear map ϕ is an isometry for $\|\cdot\|_{op}$ if and only if ϕ^* is an isometry for $\|\cdot\|_{tr}$.

Isometries for Ky Fan k -norms on $M_{m,n}$

Theorem [Grone and Marcus 1977, Li and Tsing 1988]

Let $\phi : M_{m,n} \rightarrow M_{m,n}$ be linear, and let $k \leq m$. TFAE

- (a) ϕ is an isometry for F_k .
- (b) $\phi^*(\mathcal{R}_k) = \mathcal{R}_k$, where \mathcal{R}_k consists of matrices in $M_{m,n}$ such that $s_1(X) = \cdots = s_k(X) = 1$, $s_{k+1}(X) = 0$.
- (c) There are $U \in \mathcal{U}_m$, $V \in \mathcal{U}_n$ such that one of the following holds.
 - (c.1) ϕ has the form $A \mapsto UAV$.
 - (c.2) $m = n$ and ϕ has the form $A \mapsto UA^tV$.
 - (c.3) $(\mathbb{F}, m, n, k) = (\mathbb{R}, 4, 4, 2)$

$$A \mapsto U[A - 2\psi(A)]V \text{ or } A \mapsto U[A - 2\psi(A)]^tV.$$

Here ψ is an orthogonal projection of A onto a four dimensional subspace of $M_4(\mathbb{R})$.

Additional results on $M_{m,n}$

Theorem [Li and Tsing 1990, Djokovic 1994]

Let $\phi : M_{m,n} \rightarrow M_{m,n}$ be a linear isometry for a UI norm $\|\cdot\|$. Then there are $U \in \mathcal{U}_m$, $V \in \mathcal{U}_n$ such that one of the following holds.

- (1) ϕ has the form $A \mapsto UAV$.
- (2) $m = n$ and ϕ has the form $A \mapsto UA^tV$.
- (3) $\|A\| = \gamma(\operatorname{tr} AA^*)^{1/2}$ and ϕ is unitary.
- (4) $\|A - 2\psi(A)\| = \|A\|$ for all $A \in M_4(\mathbb{R}) = M_{m,n}$ and

$$A \mapsto U[A - 2\psi(A)]V \text{ or } A \mapsto U[A - 2\psi(A)]^tV.$$

Proof. By differential geometry in [LT] and group theory in [DL].

Theorem [Li, 1986] For $\mathbb{F} = \mathbb{C}$, a mapping $\phi : M_{m,n} \rightarrow M_{m,n}$ satisfies

$$F_k(\phi(A) - \phi(B)) = F_k(A - B) \quad \text{for all } A, B \in M_{m,n}$$

if and only if there are $U \in \mathcal{U}_m$, $V \in \mathcal{U}_n$, $Z \in M_{m,n}$ such that one of the following holds.

(a) ϕ has the form

$$X \mapsto UXV + Z \quad \text{or} \quad X \mapsto U\bar{X}V + Z.$$

(b) $m = n$ and ϕ has the form

$$X \mapsto UX^tV + Z \quad \text{or} \quad X \mapsto UX^*V + Z.$$

Idea of proof.

By Ulam's Theorem, $A \mapsto \phi(A) - \phi(0)$ is real linear.

Characterize real linear isometries.

Note

We [Chan, Li, Sze] are currently extending the result to general UI norms.

What next?

Block triangular matrices

Let $T(n_1, \dots, n_r)$ be the set of upper triangular block matrices

$$(A_{ij})_{1 \leq i, j \leq r}$$

with $A_{jj} \in M_{n_j}$.

When $r = n$ and $n_j = 1$ for all j , we get the algebra of upper triangular matrices.

Up to unitary similarity, these are all the finite dimensional nest algebras.

Theorem [Li, Šemrl, Sourour, 2003/04] A surjective linear map

$$\phi : T(n_1, \dots, n_r) \rightarrow T(m_1, \dots, m_s)$$

preserves the Ky Fan k -norm if and only if $r = s$ and there exist $U, V \in T(m_1, \dots, m_s)$ such that one of the following holds.

- (a) $(n_1, \dots, n_r) = (m_1, \dots, m_r)$ and $A \mapsto UAV$.
- (b) $(n_1, \dots, n_r) = (m_r, \dots, m_1)$ and $A \mapsto UA^+V$.

Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}^+ = \begin{pmatrix} a_{33} & a_{23} & a_{13} \\ 0 & a_{22} & a_{12} \\ 0 & 0 & a_{11} \end{pmatrix}$$

Idea of proof.

1. Show that if A and B has disjoint ranges and kernels then so is their images.
2. Show that rank one matrices are mapped to rank one matrices.

The operator norm again

Observation If there are $U, V \in \mathcal{U}_k$ and a contractive linear map $f : M_n \rightarrow M_{k-n}$ such that $\phi : M_n \rightarrow M_k$ has the form

$$A \mapsto U[A \oplus f(A)]V \quad \text{or} \quad A \mapsto U[A^t \oplus f(A)]V,$$

then

$$\|\phi(A)\|_{op} = \|A\|_{op} \quad \text{for all } A \in M_n.$$

Question of Ngai-Ching Wong. Does the converse hold?

More generally, ϕ may have the form

$$A \mapsto U[\underbrace{A \oplus \dots \oplus A}_p \oplus \underbrace{A^t \oplus \dots \oplus A^t}_q \oplus f(A)]V.$$

Theorem [Cheung, Li, Poon, 2003/4] Suppose $\mathbb{F} = \mathbb{C}$. The converse holds if $k \leq 2n - 1$; otherwise, there are other isometries. For example, if ϕ has the form

$$A \mapsto P^* \left[\underbrace{A \oplus \cdots \oplus A}_p \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_q \oplus f(A) \right] Q,$$

for some suitable (high rank relative to k and $p + q$) partial isometries P and Q , then ϕ is an isometry for $\|\cdot\|_{op}$.

Question: Is that all?

Higher Ky Fan norms

Theorem [Li, Poon, Sze, 2003/4] Suppose $\mathbb{F} = \mathbb{C}$. Let $k, k' \in \mathbb{N}$ with $k' > 1$, and $\phi : M_{m,n} \rightarrow M_{r,s}$ be linear. Then

$$F_k(\phi(A)) = F_{k'}(A) \quad \text{for all } A \in M_{m,n}$$

iff there are $p, q \in \mathbb{N}$ and partial isometries P and Q such that:

(a) $k' < m$, $k'(p + q) = k$, and ϕ has the form

$$A \mapsto (p + q)^{-1} P^* \left[\underbrace{A \oplus \cdots \oplus A}_p \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_q \right] Q \text{ or}$$

(b) $k' = m$, $k'(p + q) \leq k$, and ϕ has the form

$$A \mapsto P^* [(r_1 A \oplus \cdots \oplus r_p A) \oplus (s_1 A^t \oplus \cdots \oplus s_q A^t)] Q,$$

where $r_i > 0$, $s_i > 0$ and $r_1 + \cdots + r_p + s_1 + \cdots + s_q = 1$.

Further research

Characterize UI norms on other spaces or algebras.

- * [Chan, Li, Tu, 2001] c -spectral isometries on $B(H)$.
- * [Cui, Li, Hou] are working on (surjective) Ky Fan norm isometries from a nest algebra in $B(H)$ to a nest algebra in $B(K)$.
- * One may consider Ky Fan norm isometries from on block triangular algebras without the surjective assumption.

Characterize isometries for other classes of norms.

- * [Li and Tsing, 1990] USI norm isometries on H_n .
- * [Guralnick and Li, 1997] USI/UCI norm isometries on M_n .

Characterize isometries for norms on matrices over other fields and skew-fields (with certain evaluations/norms)?

Hope to hear answers from you soon,
or I can tell you more next time.

THANK YOU FOR YOUR ATTENTION!