Recent Results on Spectrum Preserving Maps

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Characterize $\phi : \mathcal{M} \to \mathcal{M}$ with some special properties:

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Very often, ϕ is assumed to be linear, additive, multiplicative, analytic, injective, surjective, unital ....

Some classical results

Theorem [Frobenius, 1897] A linear operator $\phi : M_n \to M_n$ satisfies

$$\det(A) = \det(\phi(A)) \qquad \text{for all } A \in M_n$$

if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that ϕ has the form

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Theorem [Dieudonné, 1949] A bijective linear map $\phi : M_n \to M_n$ mapping the set of singular matrices into itself has the form

$$A \mapsto MAN \qquad \text{or} \qquad A \mapsto MA^tN$$

for some $M, N \in M_n$ with $\det(MN) \neq 0$.

Theorem [Hua, 1951] A bijective map $\phi : M_n \to M_n$ satisfies the condition that

$$\text{rank}\,(\phi(A) - \phi(B)) = 1 \iff \text{rank}\,(A - B) = 1$$

if and only if there are $M, N, R \in M_n$ with $\det(MN) \neq 0$ and a complex field automorphism σ such that ϕ has the form

$$(a_{ij}) \mapsto M(\sigma(a_{ij}))N + R \quad \text{or} \quad (a_{ij}) \mapsto M(\sigma(a_{ij}))^t N + R.$$

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Theorem [Marcus & Moyls, 1959]

Linear preservers of eigenvalues/spectrum on matrices have the standard form

$$T(A) = S^{-1}AS \qquad \text{for all } A \in M_n \tag{S1}$$

or

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Theorem [Jafarian & Sourour, 1986] (see also [Šemrl, 2002])

The same conclusion holds for surjective linear preservers of spectrum on $B(X)$.

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Theorem [Bai and Hou, 2003]
Suppose $\mathcal{M}_j$ is a standard operator algebra in $B(X_j)$ for $j = 1, 2$. If $T : \mathcal{M}_1 \to \mathcal{M}_2$ is surjective such that

$$\mathrm{Sp}\,(A + \mu B) = \mathrm{Sp}\,(T(A) + \mu T(B))$$

for all $\mu \in \{1, 2\}$ and $A, B \in \mathcal{M}_1$, then T is linear and has the standard form.

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for all $\mu \in \{1, 2\}$ and $A, B \in \mathcal{M}_1$, then T is linear and has the standard form.

The function Sp can be replaced by point spectrum, continuous spectrum, left invertible spectrum, etc.

Theorem [Hochwald, 1994], [Cheung, Fallat and Li, 2002]
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Theorem [Baribeau and Ransford, 2000]
Continuously differentiable preservers of spectra on M_n are local automorphisms, i.e.,

$$T(A) = S_A^{-1}AS_A.$$

So, T preserves the Jordan forms of matrices. Here S_A depends on A, and the strategic choice of S_A results in a continuous map.

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Then T preserves the spectrum/eigenvalues/Jordan forms of matrices as long as T maps each equivalence class into itself.

We need to impose some mild assumptions to get nice theorems!

Theorem [Molnár, 2001] Let $\mathcal{M} = M_n$ or $B(X)$, and let $T : \mathcal{M} \to \mathcal{M}$ be a surjective map satisfies

$$\mathrm{Sp}\,(AB) = \mathrm{Sp}\,(T(A)T(B)) \qquad \text{for all } A, B \in \mathcal{M}.$$

Then there is an invertible S and with $\xi \in \{-1, 1\}$ such that

$$T(A) = \xi S^{-1}AS \qquad \text{for all } A \in \mathcal{M} \tag{$S1'$}$$

or

$$T(A) = \xi S^{-1}A^tS \qquad \text{for all } A \in \mathcal{M}. \tag{$S2'$}$$

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Note that the (modified) standard form (S2') is now admissible.

Theorem [Hou and Huang, 2005] Let $\mathcal{M} = M_n$ or a standard operator algebra in $B(X)$ and let $T : \mathcal{M} \to \mathcal{M}$ be a surjective map satisfies

$$\text{Sp}\,(ABA) = \text{Sp}\,(T(A)T(B)T(A)) \qquad \text{for all } A, B \in \mathcal{M}$$

has the form (S1') or (S2') for some scalar ξ with $\xi^3 = 1$.

Theorem [Chan, Li, Sze, June 2005] Let $\mathcal{M} = M_n$ or H_n, and let $A * B$ denote the product

(i) AB, (ii) ABA, or (iii) $AB + BA$.

A map $T : \mathcal{M} \to \mathcal{M}$ satisfies

$$\mathrm{Sp}\,(A * B) = \mathrm{Sp}\,(T(A) * T(B)) \qquad \text{for all } A, B \in \mathcal{M}$$

if and only if there is an invertible (respectively, unitary) S such that T has the form

$$T(A) = \xi S^{-1} A S \qquad \text{for all } A \in \mathcal{M},$$

or

$$T(A) = \xi S^{-1} A^t S \qquad \text{for all } A \in \mathcal{M},$$

where $\xi^2 = 1$ for cases (i) and (iii), and $\xi^3 = 1$ for case (ii).

Idea of proof.

1. If T preserves eigenvalues, then it satisfies

$$\text{tr}\,(T(A)T(B)) = \text{tr}\,(AB) \text{ for all } A, B,$$

which implies that T is invertible linear, and has the standard form.

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3. Show that $L_A = L_B$ for all $A, B \in \mathcal{S}$. So, $T = L$ on $\mathcal{S}$.

4. Modify T and L so that $L(X) = X$ for all $X \in \mathcal{S}$. Then use the fact that

$$\text{Sp}\,(T(X)T(A)) = \text{Sp}\,(XA) = \text{Sp}\,(L(X)A) = \text{Sp}\,(T(X)A)$$

for all $A \in \mathcal{M}$ and $X \in \mathcal{S}$ to show $T(A) = A$ for all $A \in \mathcal{M}$.

Theorem [CLS, June 2005] Suppose $\mathcal{M} = M_n$ or H_n. Let $(i_1, \ldots, i_m)$ be such that $\{i_1, \ldots, i_m\} = \{1, \ldots, k\}$ and there is a (special) i_r which is different from all other i_s. Define

$$A_1 * \cdots * A_k = A_{i_1} \cdots A_{i_m}$$

or

$$A_1 * \cdots * A_k = A_{i_1} \cdots A_{i_m} + A_{i_m} \cdots A_{i_1}.$$

If $T : \mathcal{M} \to \mathcal{M}$ satisfies

$$\mathrm{Sp}\,(T(A_1) * \cdots * T(A_k)) = \mathrm{Sp}\,(A_1 * \cdots * A_k) \quad \forall A_1, \ldots, A_k \in \mathcal{M},$$

then there exist an invertible/unitary S and a scalar ξ with $\xi^m = 1$ such that

$$T(A) = \xi S^{-1} A S \qquad \text{for all } A \in \mathcal{M} \tag{J1}$$

or

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The form (J2) can happen if and only if

$$(i_1, \ldots, i_{r-1}, i_{r+1} \cdots, i_m) = (i_m, \ldots, i_{r+1}, i_r, \ldots, i_1).$$

Example Suppose $A * B = AABB$. Then we can define T so that $T(X) = I$ whenever $X^2 = I$, and $T(X) = X$ otherwise.

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Extension to the infinite dimensional case

Theorem [Hou and Li, November 2005] Let $\mathcal{M}_i$ be a standard operator algebra of $B(X_i)$ or the real space of self-adjoint operators in $B(H_i)$ for $i = 1, 2$. If $T : \mathcal{M}_1 \to \mathcal{M}_2$ is a surjective map satisfying

$$\mathrm{Sp}\,(T(A_1) * \cdots * T(A_k)) = \mathrm{Sp}\,(A_1 * \cdots * A_k)$$

for all $A_1, \ldots, A_k \in \mathcal{M}_1$ such that $\mathrm{rank}(A_1 * \cdots * A_k) \leq 2$, then T has the standard form (J1) and (J2), i.e., T is a unit multiple of a Jordan isomorphism.

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Note that (J2) can only happen when X_1 and X_2 are reflexive.

Idea of proof.

1. Reduce to the special cases of $A * B = A^r B A^s$ or $A^r B A^s + A^s B A^r$.

2. Show that the images of rank one operators have nice form.

3. Use the result on rank one operators to show that the images of other operators also behave well.

4. Deduce the restriction on X_1, X_2 if (J2) holds, and deduce the condition on $(j_1, \ldots, j_m)$ in the general case.

Further research

1. Can we get the conclusion of Hou and Li if

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for all $A_1,\ldots,A_k \in \mathcal{M}_1$ such that $\mathrm{rank}(A_1*\cdots*A_k) \leq 1$?

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5. How about the problems on triangular matrices, nest algebras, general linear groups, or other semi-groups of matrices or operators?

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