Upper bounds for the norm of the sum of operators and related results

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Basic question and result

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$$\|A + B\| \leq \|A + \mu I\| + \|B - \mu I\|, \qquad \mu \in \mathbb{C}.$$

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Question How good is this estimate?

Answer This is best possible if there is no further information of A and B is given because

$$\sup_{U,V \text{ unitary}} \|U^*AU + V^*BV\| = \min_{\mu\in\mathbb{C}}\{\|A+\mu I\| + \|B-\mu I\|\}.$$

Main theorem

For any $T \in B(H)$, let

$$\Omega(T) = \{(a, |b|) : Tx = ax + b\tilde{x} \text{ for some } x, \tilde{x} \in H$$
$$\text{with } (x, x) = (\tilde{x}, \tilde{x}) = 1,\ (x, \tilde{x}) = 0\}.$$

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$$\sup_{U,V \text{ unitary}} \|U^*AU + V^*BV\| = \min_{\mu \in \mathbb{C}} \{\|A + \mu I\| + \|B - \mu I\|\},$$

which is the same as

$$\sup\{\|AX + XB\| : X \in B(H), \|X\| \leq 1\}$$

and

$$\sup\{\|u + v\| : u \in \Omega(A),\ v \in \Omega(B)\}.$$

Idea of proof

If $Ax = u_1 x + u_2 \tilde{x}$ and $By = v_1 y + v_2 \tilde{y}$, then there are unitary U and V such that

$$U^* A U e_1 = u_1 e_1 + |u_2| e_2 \quad \text{and} \quad V^* B V e_1 = v_1 e_1 + |v_2| e_2.$$

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So,

$$\|u + v\| = \|(U^* A U + V^* B V) e_1\| \le \|U^* A U + V^* B V\|$$

and

$$\sup_{u \in \Omega(A), v \in \Omega(B)} \|u + v\| \le \sup_{U,V \text{ unitary}} \|U^* A U + V^* B V\|.$$

Also, we have

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And

$$\begin{aligned} \|AX + XB\| &= \|(A+\mu I)X + (B-\mu I)X\| \\ &\leq \|(A+\mu I)X\| + \|(B-\mu I)X\| \\ &\leq \|A+\mu I\| + \|B-\mu I\| \end{aligned}$$

implies that

$$\sup_{\|X\|\leq 1} \|AX + XB\| \leq \min_{\mu\in\mathbb{C}} \{\|A+\mu I\| + \|B-\mu I\|\}.$$

It remains to show that

$$\min_{\mu \in \mathbb{C}} \{\|A + \mu I\| + \|B - \mu I\|\} \leq \sup_{u \in \Omega(A), v \in \Omega(B)} \|u + v\|.$$

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Case 1 $\dim H < \infty$.

$$\|A\|+\|B\| = \min_{\mu\in\mathbb{C}}\{\|A+\mu I\|+\|B-\mu I\|\}$$

if and only if there are unit vectors $x, y \in H$ such that

$$\|Ax\| = \|A\|, \quad \|By\| = \|B\|, \qquad (Ax,x)/\|A\| = (By,y)/\|B\|.$$

Then, $Ax = u_1 x + u_2 \tilde{x}$ and $By = v_1 y + v_2 \tilde{y}$ such that

$$\|u+v\| = \|u\|+\|v\| = \|A\|+\|B\|.$$

Case 2 dim$H = \infty$.

Assume that

$$\min_{\mu\in\mathbb{C}}\{\|A+\mu I\| + \|B-\mu I\|\} - \varepsilon > \sup_{u\in\Omega(A),v\in\Omega(B)} \|u+v\|.$$

Construct finite dimensional compressions $\tilde{A}$ and $\tilde{B}$ of A and B so that

$$\|\tilde{A}+\mu^* I\| + \|\tilde{B}-\mu^* I\| = \|u^*+v^*\| \le \sup_{u\in\Omega(A),v\in\Omega(B)} \|u+v\|,$$

but

$$\|\tilde{A}+\mu^* I\| + \|\tilde{B}-\mu^* I\| \ge \min_{\mu\in\mathbb{C}}\{\|A+\mu I\| + \|B-\mu I\|\} - \varepsilon,$$

which is a contradiction.

Some consequences

Corollary Let $A, B \in B(H)$. Then

$$\|A\| + \|B\| = \min_{\mu \in \mathbb{C}} \{\|A + \mu I\| + \|B - \mu I\|\}$$

if and only if there is a sequence of unitary operators $U_1, U_2, \ldots,$ such that

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Corollary Let $A, B \in B(H)$. Then

$$\|A + B\| = \sup_{U,V \text{ unitary}} \|U^* A U + V^* B V\|$$

if and only if there is $\mu^* \in \mathbb{C}$ such that

$$\|A + \mu^* I\| + \|B - \mu^* I\| = \|A\| + \|B\|.$$

Theorem Suppose $A, B \in B(H)$. Then

$$\sup_{U,V \text{ unitary}} \|U^*AU - V^*BV\| = \min_{\mu \in \mathbb{C}} \{\|A - \mu I\| + \|B - \mu I\|\},$$

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The last quantity is the operator norm of the derivation operator

$$X \mapsto AX - XB,$$

which was studied by Stampfli.

Theorem Let $A \in B(H)$. Define

$$\begin{aligned} g_A(t) &= \sup_{U,V \text{ unitary}} \|U^*AU - e^{it}V^*AV\| \\ &= \min_{\mu\in\mathbb{C}}\{\|A+\mu I\| + \|A-\mu e^{-it}I\|\} \\ &= \min_{\mu\in\mathbb{C}}\{\|A+\mu e^{it/2}I\| + \|A-\mu e^{-it/2}I\|\}. \end{aligned}$$

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Then $g_A(t) = g_A(-t)$ and is monotonic increasing in $[0, \pi]$.

In particular, for each $t \in [0, \pi]$,

$$2\|A\| = g_A(\pi) \geq g_A(t) \geq g_A(0) = 2\min_{\mu\in\mathbb{C}} \|A - \mu I\|.$$

Spectral circles and dilations

For $\mu \in \mathbb{C}$ and $r > 0$ let $\Gamma(\mu; r) = \{z \in \mathbb{C} : |z - \mu| = r\}$.

von Neumann inequality Let f be a polynomial.

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Note that for every $A \in B(H)$, there is a unique choice of μ^* such that

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Theorem Let $A, B \in B(H)$, and μ^* be such that

$$\|A - \mu^* I\| + \|B - \mu^* I\| \leq \|A - \mu I\| + \|B - \mu I\| \qquad \text{for all } \mu \in \mathbb{C}.$$

Set $r_1 = \|A - \mu^* I\|$ and $r_2 = \|B - \mu^* I\|$. Then for any U, V and f, g,

$$\|U^* f(A) U + V^* g(B) V\| \leq \max_{z \in \Gamma(\mu^*; r_1)} |f(z)| + \max_{z \in \Gamma(\mu^*; r_2)} |g(z)|.$$

Theorem Suppose $A, B \in B(H)$. Then

$$\sup_{U,V \text{ unitary}} \|U^*AU - V^*BV\| = \min_{\tilde{A},\tilde{B}} \sup_{\tilde{U},\tilde{V} \text{ unitary}} \|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\|,$$

where $\tilde{A}, \tilde{B}$ range over all possible normal dilations of A, B on a common Hilbert space.

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where $\tilde{A}, \tilde{B}$ range over all possible normal dilations of A, B on a common Hilbert space.

Actually, one may choose $\tilde{A}, \tilde{B}$ with spectra $\sigma(\tilde{A})$ and $\sigma(\tilde{B})$ enclosed by $\Gamma(\mu^*; r_1)$ and $\Gamma(\mu^*; r_2)$, and such a choice of $(\tilde{A}, \tilde{B})$ always yields

$$r_1 + r_2 = \sup_{U,V} \|U^*AU - V^*BV\| = \sup_{\tilde{U},\tilde{V}} \|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\|.$$

Computation of optimal values

Proposition If $A, B \in B(H)$ are normal with given spectra, then the optimal values of $\sup_{U,V} \|U^*AU + V^*BV\|$ can be determined by solving

$$\min_{\mu \in \mathbb{C}} \max\{|\alpha + \mu| + |\beta - \mu| : \alpha \in \sigma(A), \beta \in \sigma(B)\}.$$

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In general, one may construct the sets $\Omega(A)$ and $\Omega(B)$ in $\mathbb{R}^3$ to determine

$$\sup\{\|u + v\| : u \in \Omega(A), v \in \Omega(B)\}.$$

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Recall that

$$w(T) = \sup\{|(Tx, x)| : x \in H, (x, x) = 1\}.$$

Theorem Let $A, B \in B(H)$. If

$$\min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbb{C}\}$$

is attained at μ^*, then

$$|\mu^*| \leq \max\{w(A), w(B)\} \leq \max\{\|A\|, \|B\|\}.$$

Characterization of operator norm on M_n

For any norm ν on $B(H)$, we have

$$\sup_{U,V} \nu(U^*AU + V^*BV) \leq \min_{\mu \in \mathbb{C}} \{\nu(A + \mu I) + \nu(B - \mu I)\}.$$

The equality holds if ν is a multiple of the operator norm.

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Example For $p \geq 1$, the Schatten p-norm defined by

$$\|A\|_p = \left\{\sum s_j(A)^p\right\}^{1/p}$$

on a symmetrically norm ideal of compact operators in $B(H)$.

Theorem Suppose ν is a unitarily invariant norm on M_n. The following conditions are equivalent.

(a) For all $A, B \in M_n$,

$$\sup_{U,V} \nu(U^*AU + V^*BV) = \min_{\mu \in \mathbb{C}} \{\nu(A + \mu I) + \nu(B - \mu I)\}. \qquad (1)$$

(b) Equation (1) holds for

$A = E_{12} + E_{23} + \cdots + E_{n1}$ and

$B = e^{it}(E_{11} - E_{22})$ for any t such that $e^{i2nt} \neq 1$.

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Question Can we extend the result to other class of norms?

Related questions

Let $A, B \in B(H)$ equipped with a (unitarily invariant) norm ν. Determine

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Determine

$$\sup_{A,B\neq 0} \nu(AB - BA)/\nu(A)\nu(B).$$

Fact For $\nu(X) = \|X\|$, the supremum is 2.

Conjecture For $\nu(X) = [\mathrm{tr}\,(X^*X)]^{1/2}$ on M_n, the supremum is $\sqrt{2}$.

Thank you for your attention!