1. A 1960 issue of Science magazine included an article by von Foerster and his colleagues P. M. Mora and L. W. Amiot proposing a formula representing a best fit to available historical data on world population; the authors then predicted future population growth on the basis of this formula. The formula gave 2.7 billion as the 1960 world population and predicted that population growth would become infinite by Friday, November 13, 2026 - von Foerster’s 115th birthday anniversary - a prediction that earned it the name “the Doom’s Day Equation.”

(a) The doom’s day equation which von Foerster et.al. proposed is \( \frac{dN}{dt} = kN^2 \) where \( N(t) \) is the population at time \( t \) and \( k > 0 \). Solve the equation to express \( N \) as a function of \( t \) and \( N(0) = N_0 \).

(b) Find the time \( t_* \) when the solution tends to infinity. Assuming that \( t = 0 \) is the year 1960 and \( N_0 = 2.7 \) billion, and \( t_* = 66 \) (the year 2026), find the value of \( k \). What is the dimension and unit of \( k \)?

(c) Use the value of \( k \) solved in part (b) and \( N_0 = 2.7 \) billion to find the population today (year 2014). How does this estimate compare to the real population today? (Google to find out the population today)

2. Consider a population model:
\[
\frac{dx}{dt} = kx \left( 1 - \frac{x}{N} \right) - M, \quad x(0) = x_0,
\]
where \( k, M, N, x_0 \) are positive parameters.

(a) In the following table, fill in the dimensions of all parameters in terms of the dimensions of variables.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension ( \tau )</th>
<th>Parameter</th>
<th>Dimension ( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( \lambda )</td>
<td>( M )</td>
<td>( \hat{N} )</td>
</tr>
<tr>
<td>( x_0 )</td>
<td></td>
<td>( \hat{N} )</td>
<td></td>
</tr>
</tbody>
</table>

(b) Use the change of variable:
\[ y = \frac{x}{N}; \quad s = kt. \]

Derive the new equation (including the initial condition) in the new variables \( y \) and \( s \).

3. Suppose that \( N(t) \) denotes the size of a population at time \( t \). The population evolves according to the logistic equation but, in addition, predation reduces the size of the population so that the rate of change is given by
\[
\frac{dN}{dt} = N \left( 1 - \frac{N}{50} \right) - \frac{9N}{5 + N}.
\]
(a) Find the equilibrium points of the equation.
(b) Determine the stability of the equilibrium points by using linearization.
(c) Sketch the phase line. If $N(0) = 35$, what is the limit of $N(t)$ as $t \to \infty$?


(a) You do not explain this part, but you will use the equation in following parts.
(b) Use a table as in problem 2 to list the dimensions of all parameters and variables.
(c) Use change of variables: $t^* = \mu t$, $N^* = \frac{\alpha}{C_0} N$, and $C^* = \frac{1}{k_n} C$ to obtain a new equation with dimensionless equations of $N^*$ and $C^*$.
(d) Find all steady states of the dimensionless equation from (c).
(e) We will not do this part in this assignment.