Count Assignment 7
Math 345, Prof. Shi

Due: Friday, Oct 31 (5pm)

1. (Page 362 problem 2, modified) Consider the following system of equations (Lefschetz, 1977):

\[ \begin{align*}
    x' &= y = f(x, y), \\
    y' &= x(a^2 - x^2) + by = g(x, y), \quad a \neq 0, b \neq 0.
\end{align*} \]

(a) Show that the equilibrium points are (0, 0), (a, 0), and (a, 0).

(b) Evaluate the expression \( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \) for \((x, y) \in \mathbb{R}^2\).

(c) Use Bendixson’s criterion to rule out the presence of limit-cycle solutions about each one of these equilibria.

(d) Use \texttt{pplane} to plot the phase portrait for \( a = 2 \) and \( b = 0 \). Is there a limit-cycle? Explain why the Bendixson’s criterion is not applicable for this case.

2. (Page 363 problem 6, modified) Consider the following system of equations (Odell, 1980), which are said to describe a predator-prey system:

\[ \begin{align*}
    x' &= x[x(1 - x) - y], \\
    y' &= y(x - a).
\end{align*} \]

(a) Find and sketch the nullclines of the system, and mark the direction of the equation vector field on the nullclines. (consider two cases: \( a > 1 \) and \( a < 1 \))

(b) Find all equilibrium points of the system.

(c) At each equilibrium point, linearize the system and identify the type of the linearized system. (stable node, unstable node, stable spiral, unstable spiral, saddle, or cannot be linearly determined etc.)

(d) Use \( a \) as a bifurcation parameter, and show that a Hopf bifurcation occurs at the positive equilibrium for some \( a = a_1 > 0 \).

(e) Use \texttt{pplane} to plot the phase portrait for \( a < a_1 \) and \( a > a_1 \). What is the behavior of the system? Describe asymptotic behavior for typical initial values.

3. Consider an SIS epidemic model: \( S \rightarrow I \rightarrow S \), which can be described by the system of equations:

\[ \begin{align*}
    \frac{dS}{dt} &= -\beta SI + \alpha I, \\
    \frac{dI}{dt} &= \beta SI - \alpha I.
\end{align*} \]

Here \( \beta \) and \( \alpha \) have the same meaning as in the SIR model. Let the total population be \( N \).

(a) Use change of variables: \( u = \frac{S}{N}, v = \frac{I}{N}, s = \alpha t \) to obtain dimensionless system:

\[ \begin{align*}
    \frac{du}{ds} &= -R_0uv + v, \\
    \frac{dv}{ds} &= R_0uv - v.
\end{align*} \]

Here \( R_0 \) is the new parameter. Find the expression of \( R_0 \) in terms of \( \alpha, \beta \) and \( N \).

(b) Prove that \( u(s) + v(s) = 1 \) for solution \((u(s), v(s))\) of the new system.

(c) Find all equilibrium points of the system and determine their stability. Notice that any equilibrium \((u, v)\) satisfies \( 0 \leq u, v \leq 1 \) and \( u + v = 1 \).

(d) Use \texttt{pplane} to plot the phase portrait for \( R_0 < 1 \) and \( R_0 > 1 \).