First epidemic model: SIR model(Kermack-Mckendrick, 1927)

$$S' = -\beta SI, \;\; I' = \beta SI - \alpha I, \;\; R' = \alpha I$$

Assumptions

1. Total population is a constant N

2. A average infective makes contact sufficient to transmit infection with βN others per unit time

3. A fraction α of infectives leave the infective class per unit time (or the average time of recovering from the disease is $1/\alpha$)

Basic reproduction number: $R_0 = \dfrac{\beta N}{\alpha}$

(**Not** $\dfrac{\beta S(0)}{\alpha}$ used in last lecture)

Example: A disease is introduced by two visitors into a town with 1200 inhabitants. An average infective is in contact with 0.4 inhabitant per day. The average duration of the infective period is 6 days, and recovered infectives are immune against reinfection. How many inhabitants would have to be immunized to avoid an epidemic?

$\beta N = 0.4$, so $\beta = 0.4/1200 = 1/3000 \approx 0.03\%$ (means an average infective is in contact with 0.03% of all inhabitants per day.
$\alpha = 1/6$ per day
Dimension of β is per person per day,
and dimension of α is per day.

$$\frac{dS}{d\tau} = -\beta SI, \quad \frac{dI}{d\tau} = \beta SI - \alpha I, \quad \frac{dR}{d\tau} = \alpha I$$

Variable	Dimension	Parameter	Dimension
t	τ	S	ρ
N	ρ	I	ρ
		R	ρ
		α	τ^{-1}
		β	$\rho^{-1}\tau^{-1}$

Nondimensionalization: $u = \frac{S}{N}$, $v = \frac{I}{N}$, $w = \frac{R}{N}$, $t = \alpha\tau$

$$\frac{du}{dt} = -R_0 uv, \; \frac{dv}{dt} = (R_0 u - 1)v, \; \frac{dw}{dt} = v$$

Qualitative analysis:

$$u' = -R_0 uv$$
$$v' = (R_0 u - 1)v$$

Behavior:

(1) If $R_0 < 1$, then $v(t)$ (thus $I(t)$) is decreasing to zero, no epidemics;

(2) If $R_0 > 1$, then $v(t)$ (thus $I(t)$) is increasing initially if $u(0) \approx 1$, and is decreasing to zero as $t \to \infty$, epidemics occurs.

$\dfrac{dv}{du} = -1 + \dfrac{1}{R_0 u}$, $v(t) + u(t) - \frac{1}{R_0} \ln u(t) = v(0) + u(0) - \frac{1}{R_0} \ln u(0)$

The final size of susceptible is u_∞: $u_\infty - \frac{1}{R_0} \ln u_\infty \approx 1$

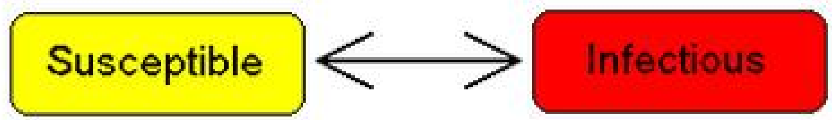
The fall of Aztecs

In the Spanish invasion of South America in 1520, the Aztecs were devastated by a smallpox epidemic introduced by one of Cortez' men. Smallpox is a lethal disease. Take $N = 1000$, $\beta = 0.1$, and the infective period for smallpox is two weeks.

1) Find R_0.
2) Compare the final size of epidemics in the two cases (i) initially the whole population is susceptible; (ii) initially population is 70% immune.

`http://www.pbs.org/conquistadors/cortes/cortes_h00.html`

SIS model: (disease with no immunity)

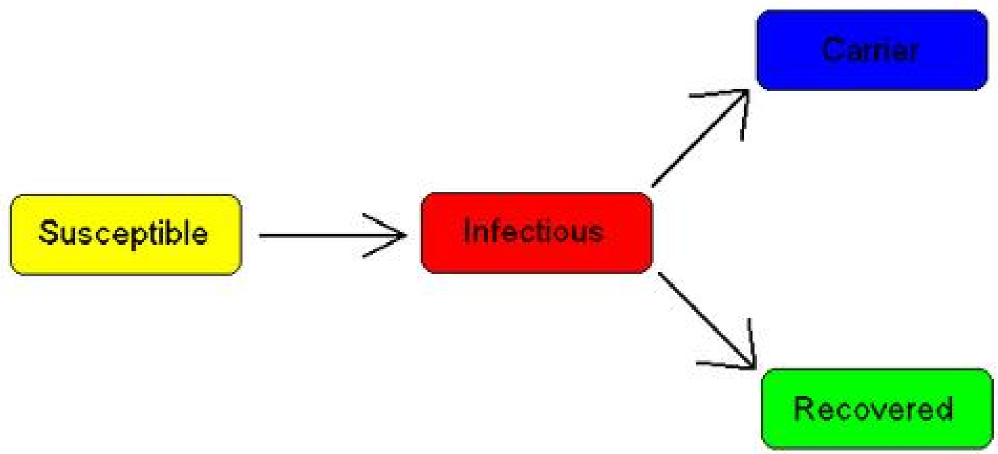


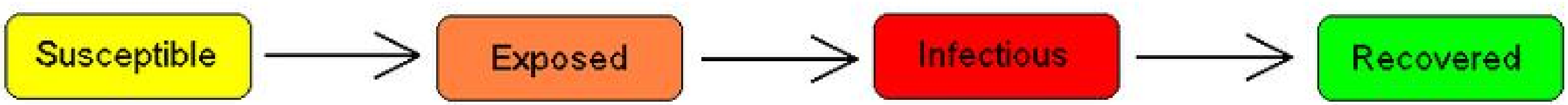
$$S' = -\beta SI + \alpha I$$
$$I' = \beta SI - \alpha I$$

Since $S + I = N$, then it can be reduced to one equation: $I' = (\beta N - \alpha)I - \beta I^2$ A logistic equation if $R_2 = \frac{\beta N}{\alpha} > 1$, in this case, $I(t)$ tends to the "carrying capacity", and an endemic occurs.

If $R_2 < 1$, then epidemic will never occur since $I(t)$ tends to 0 monotonically.

Other compartmental models:





Exposed: infected but not yet infectious;
Carrier: infectious but not sick

SIR endemic (including birth and death)

$$\frac{dS}{d\tau} = bN - \beta SI - dS, \quad \frac{dI}{d\tau} = \beta SI - \alpha I - dI - cI, \quad \frac{dR}{d\tau} = \alpha I - dR$$

b: birth rate; d: disease-unrelated death rate,
c: disease-related death rate
All new-born are in the susceptible class, so there is no vertical transmission (parent to new-born, e.g. AIDS)

Nondimensionalized version: (assuming $b = d$)
$u = \frac{S}{N}$, $v = \frac{I}{N}$, $w = \frac{R}{N}$, $t = (\alpha + b)\tau$

$$\frac{du}{dt} = \frac{b}{\alpha + b}(1 - u) - R_0 uv, \ \frac{dv}{dt} = (R_0 u - 1)v, \ \frac{dw}{dt} = \frac{\alpha}{\alpha + b}v - \frac{b}{\alpha + b}w$$

Basic reproductive number: $R_0 = \frac{\beta N}{\alpha + b}$

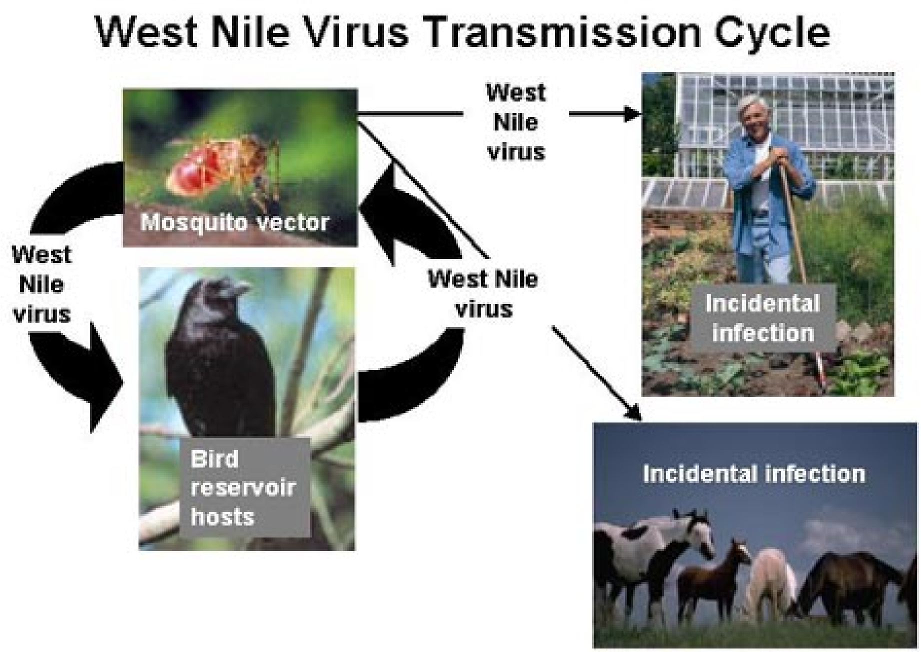
Equilibrium: $(u, v) = (1, 0)$ (disease free);
$(u, v) = (R_0^{-1}, \frac{b}{\alpha+b}(1 - R_0^{-1}))$ (endemic)

When $R_0 < 1$, the disease will die, and disease-free equilibrium is stable;
when $R_0 > 1$, the disease will stay in the population, disease-free equilibrium is unstable, and endemic equilibrium is stable.

Vector-borne infectious diseases

http://www.cdc.gov/ncidod/dvbid/

(CDC Division of Vector-Borne Infectious Diseases)



Criss-cross infections (example: malaria)

S_1: susceptible human; I_1: infected human;
S_2: susceptible mosquito; I_1: infected mosquito;
N_1: total human (constant); N_2: total mosquito (constant)

$$\frac{dS_1}{d\tau} = -ap_1\frac{S_1}{N_1}I_2 + \gamma_1 I_1,\ \frac{dI_1}{d\tau} = ap_1\frac{S_1}{N_1}I_2 - \gamma_1 I_1$$
$$\frac{dS_2}{d\tau} = -ap_2\frac{I_1}{N_1}S_2+\gamma_2 I_2+b_2N_2-d_2S_2,\ \frac{dI_2}{d\tau} = ap_2\frac{I_1}{N_1}S_2-\gamma_2 I_2-d_2I_2$$

a; biting rate of mosquito, p_1, p_2: probability of infection

West Niles Transmission

http://dx.doi.org/10.1016/j.bulm.2005.01.002

A mathematical model for assessing control strategies against West Nile virus

C. Bowmana, b, A.B. Gumelc, Corresponding Author Contact Information, E-mail The Corresponding Author, P. van den Driessched, J. Wue and H. Zhue

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