

Reaction-diffusion equation:

Example 1 Genetics drifting model (Lecture 20)

$$\frac{dp}{dt} = sp(1 - p)$$

where $p(t)$ is the fraction of one Allele at generation t

Now consider a species randomly dispersing in an unbounded habitat, and we look for one particular gene. Let $p(x, t)$ be the density of the species which possesses an advantageous allele. Then (Fisher equation or diffusive logistic equation, 1937)

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2} + sp(1 - p)$$

Phenomenon: the advantageous gene initializes at one location, and it will spread to other places.

Example 3 Epidemic model (Lecture 18-19)

$$S_t = -\beta SI$$

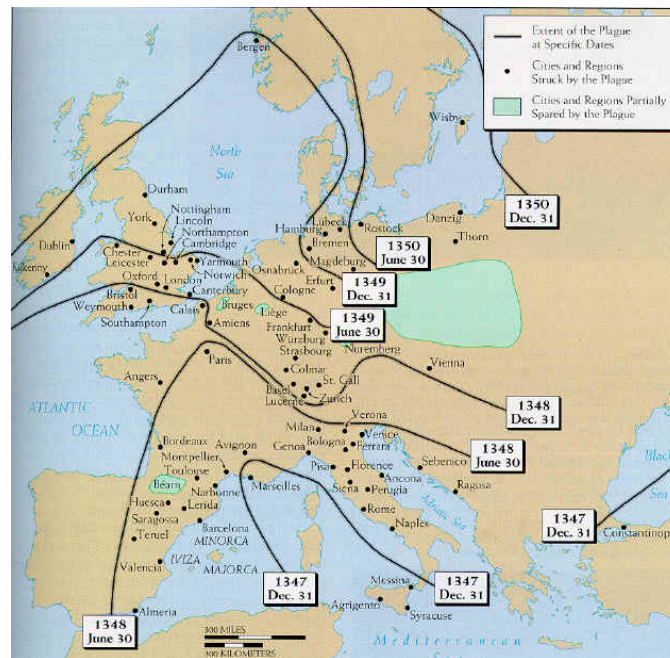
$$I_t = D_I I_{xx} + \beta SI - \alpha I$$

$$R_t = \alpha I$$

Phenomenon: The locally infected group reaches a high, then returns to zero; but nearby region will have the peak of infected people in a slightly later time; so the peak of infected propagates from center to surrounding areas. That is an epidemic wave.

Here consider the case of rabies (population of foxes). Rabies is lethal so that removed class is dead (thus no diffusion). Healthy foxes tend to stay in their own territory, but rabid will travel at random and attack other foxes. Thus only infected class will diffuse.

The Black Death, also known as the “Black Plague”, was a devastating pandemic that first struck Europe in the mid-late-14th century (1347-1350), killing between a third and two-thirds of Europe’s population. Almost simultaneous epidemics occurred across large portions of West Asia and the Middle East during the same period, indicating that the European outbreak was actually part of a multi-regional pandemic. Including Middle Eastern lands, India and China, the Black Death killed at least 75 million people.
http://en.wikipedia.org/wiki/Black_Death



Traveling wave solution: a solution with fixed spatial shape but moving when time evolves. Suppose the “shape” is $w(x)$ when $t = 0$ and the speed of moving is c , then the solution is $u(x, t) = w(x - ct)$, *i.e.* when $t = 1$, $u(x, 1) = w(x - c)$, when $t = 2$, $u(x, 2) = w(x - 2c)$, etc.

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t))$$

format of traveling wave solution: $u(x, t) = w(x - ct)$

then $w(z)$ (here we use $z = x - ct$) satisfies

$$-c \frac{dw}{dz} = \frac{d^2 w}{dz^2} + f(w), \text{ which is an ordinary differential equation.}$$

$w'' + cw' + f(w) = 0$, a 2nd order differential equation.

2nd order differential equation is a 1st order system

$$w'' + cw' + f(w) = 0$$

Let $v = w'$, then $w' = v$ and $v' = w'' = -cw' - f(w) = -cv - f(w)$

$$w' = v$$

$$v' = -cv - f(w)$$

We can do phase plane analysis!

Traveling wave solution of Fisher's equation:

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t)(1 - u(x, t)), \text{ let } u(x, t) = w(x - ct)$$
$$w'' + cw' + w(1 - w) = 0, \text{ let } v = w'$$

$$w' = v$$

$$v' = -cv - w(1 - w)$$

Equilibrium points: $(0, 0)$ and $(1, 0)$, we look for a solution satisfying $\lim_{z \rightarrow -\infty} w(z) = 0$ and $\lim_{z \rightarrow \infty} w(z) = 1$ (road to the advantageous gene).

Theorem (Fisher, 1937; Kolmogoroff-Petrovsky-Piscounoff, 1937)

Consider $\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t)(1 - u(x, t))$. For every $c \geq 2$, there is a traveling wave solution $u(x, t) = w(x - ct)$ so that $\lim_{z \rightarrow -\infty} w(z) = 0$ and $\lim_{z \rightarrow \infty} w(z) = 1$. Moreover, when the initial condition is a step function $u_0(x) = 1$ when $x < 0$ and $u_0(x) = 0$ when $x > 0$, then the solution $u(t, x)$ tends to the traveling wave solution with speed $c = 2$ (minimum speed).

Biological meaning: initially advantageous gene occupies one region $x < 0$, and the other region $x > 0$ is occupied by recessive gene. Now the evolution and diffusion (Fisher equation) pushes the advantageous gene into the recessive region, with a speed 2, and eventually the advantageous gene will occupy all regions. This is an example of **biological invasion**.

Traveling wave in epidemic model: minimal speed $c = 2\sqrt{1 - R_0^{-1}}$, so there is an epidemic wave when $R_0 > 1$.