Test 1/homework 5 (take home) solution

1. Suppose that the steady state is x, then $x = \dfrac{2x}{1+x^2}$. $x = 0$ or $1 = \dfrac{2}{1+x^2}$. The 2nd equation becomes $1 + x^2 = 2$, $x = 1$. Hence there are two steady state solutions $x = 0$ and $x = 1$. The stability of the steady state is determined by $f'(x)$ where $f(x) = \dfrac{2x}{1+x^2}$. $f'(x) = \dfrac{2-2x^2}{(1+x^2)^2}$, so $f'(0) = 2$ and $f'(1) = 0$. Therefore $x = 0$ is unstable since $|f'(0)| > 1$ and $x = 1$ is stable since $|f'(1)| < 1$. The cobwebbing shows monotone dynamics and every solution tends to the stable equilibrium $x = 1$.

2. (a)

Variable	Dimension	Parameter	Dimension
t	τ	r	τ^{-1}
N	λ	K	λ
		a	1
		b	λ^{-1}

   (b) The new equation is in a form: $\dfrac{dx}{ds} = x\left(1 - \dfrac{x}{M} - \dfrac{a}{1+x}\right)$, $x(0) = bN(0) > 0$, and $M = bK$.

   (c) Now we assume that $M = 2$, so the equation is $\dfrac{dx}{ds} = x\left(1 - \dfrac{x}{2} - \dfrac{a}{1+x}\right)$. Determine the stability of the equilibrium $x = 0$. The stability is determined by $f'(x)$. $f'(x) = 1 - x - \dfrac{a}{(1+x)^2}$ and $f'(0) = 1 - a$. Thus $x = 0$ is stable if $a > 1$ since $f'(0) = 1 - a < 0$, and $x = 0$ is unstable if $a < 1$ since $f'(0) = 1 - a > 0$.

   (d) $x = 0$ is always a steady state solution. Other steady states are given by $1 - \dfrac{x}{2} - \dfrac{a}{1+x} = 0$. Doing algebra, we find $2(1+x) - x(1+x) - 2a = 0$, $x^2 - x + 2a - 2 = 0$, thus $x = \dfrac{1 \pm \sqrt{9-8a}}{2}$. One bifurcation is $a = 9/8$ since when $a > 9/8$, the roots are complex. A saddle-node bifurcation occurs at $a = 9/8$, When $a < 9/8$, there are always two real roots: one is always positive $x_1 = \frac{1+\sqrt{9-8a}}{2}$, but the other one $x_2 = \frac{1-\sqrt{9-8a}}{2}$ could be negative this invalid for the population model. That happens when $x_2 = \frac{1-\sqrt{9-8a}}{2} = 0$, or $9 - 8a = 1$, $a = 1$. When $a < 1$, x_2 is negative. Thus we have two bifurcation points $a = 1$ and $a = 9/8$. When $0 < a < 1$, there are two (valid) equilibrium points $x = 0$ (unstable) and $x = x_1$ (stable); when $1 < a < 9/8$, there are three equilibrium points $x = 0$ (stable), $x = x_2$ (unstable) and $x = x_1$ (stable); and when $a > 9/8$, there ia only one equilibrium $x = 0$ (stable).

3. (a) $\lambda = 0.0387$, and $N_0 = 18$ according to original data and $N_0 = 15.8796$ according to best-fitting. Estimated $N(2004) \approx 205$, $N(2005) \approx 213$, and $N(2006) \approx 221$.

   (b) With $K = 400$, $\lambda = exp(0.0476) = 1.0488$ and $N_0 = 18$ according to original data and $N_0 = 14.3068$ according to best-fitting. Estimated $N(2004) \approx 185$, $N(2005) \approx 190$, and $N(2006) \approx 195$.

   With $K = 1000$, $\lambda = exp(0.0417) = 1.0426$ and $N_0 = 18$ according to original data and $N_0 = 15.3644$ according to best-fitting. Estimated $N(2004) \approx 196$, $N(2005) \approx 203$, and $N(2006) \approx 210$.

4. (10 pt)The propagation of an annual plant is governed by the second order difference equation $x_{n+2} = x_{n+1} - 6x_n$ where x_n denotes the number of plants after n-years. Find x_n given that $x_0 = 2$ and $x_1 = 4$.

   Original version: $x_{n+2} = x_{n+1} - 6x_n$, characteristic equation $\lambda^2 = \lambda - 6$, $\lambda^2 - \lambda + 6 = 0$, $\lambda = \dfrac{1 \pm \sqrt{-23}}{2}$.

   Then $x_n = c_1\left(\dfrac{1+\sqrt{-23}}{2}\right)^n + c_2\left(\dfrac{1-\sqrt{-23}}{2}\right)^n$, with $x_0 = 2$ we have $c_1 + c_2 = 2$ and with $x_1 = 4$,

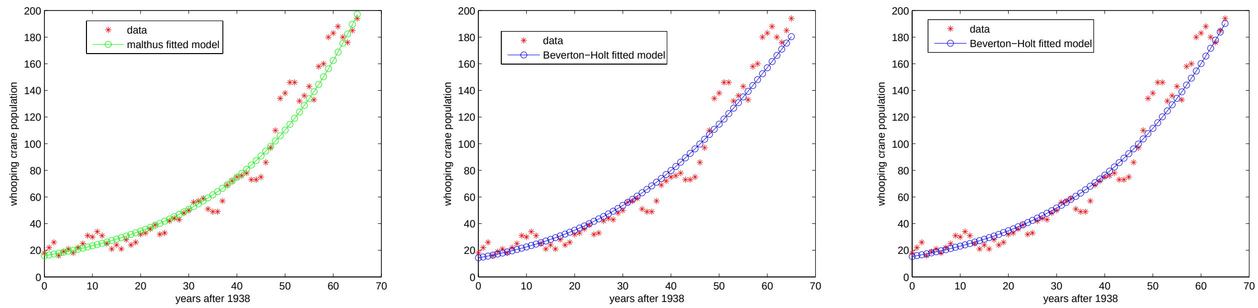

Figure 1: problem 3 (left) Malthus; (center) Beverton-Holt $K = 400$; (right) Beverton-Holt $K = 1000$

we have $c_1 \dfrac{1+\sqrt{-23}}{2} + c_2 \dfrac{1-\sqrt{-23}}{2} = 4$ we find $c_1 = 1 + \dfrac{3}{\sqrt{-23}}$ and $c_2 = 1 - \dfrac{3}{\sqrt{-23}}$

modified version: $x_{n+2} = x_{n+1} + 6x_n$, characteristic equation $\lambda^2 = \lambda + 6$, $\lambda^2 - \lambda - 6 = 0$, $\lambda = 3$ and $\lambda = -2$. Then $x_n = c_1 (3)^n + c_2 (-2)^n$, with $x_0 = 2$ we have $c_1 + c_2 = 2$ and with $x_1 = 4$, we have $3c_1 - 2c_2 = 4$ we find $c_1 = \dfrac{8}{5}$ and $c_2 = \dfrac{2}{5}$.

5. Let $J(n)$ and $A(n)$ be the populations of juvenile and adult at time n respectively. Then $A(n+1) = 0.55J(n) + 0.55A(n)$, $J(n+1) = 2 \cdot 0.55A(n)$. Or in matrix equation format: $\begin{pmatrix} J_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1.1 \\ 0.55 & 0.55 \end{pmatrix} \begin{pmatrix} J_n \\ M_n \end{pmatrix}$

6. (a) see below. (b) $\lambda_1 = 1.0254$, and eigenvector $[0.0663, 0.5672, 0.5795, 0.5815]$. (c) increases by about 2.54%. (d) Juvenile: $\dfrac{x_2}{x_1+x_2+x_3+x_4} = \dfrac{0.5672}{0.0663+0.5672+0.5795+0.5815} \approx 0.3161 = 31.61\%$; mature females: $\dfrac{x_3}{x_1+x_2+x_3+x_4} = \dfrac{0.5795}{0.0663+0.5672+0.5795+0.5815} \approx 0.3229 = 32.29\%$; (e) Suppose that the juveniles experiences a 10% decrease in survivorship (the matrix entry a_{21} becomes $0.9775 * 0.9$, and a_{22} becomes $0.9111 * 0.9$). Then the dominant eigenvalue becomes 0.9961, thus an increase turns into a decrease of 0.39%

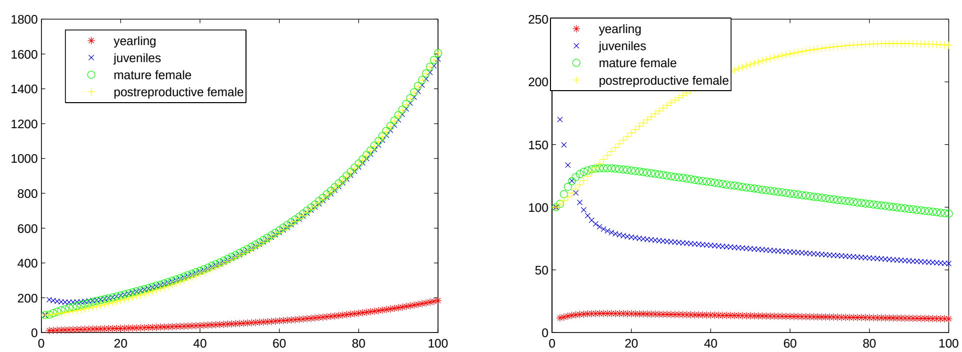

Figure 2: problem 6(left) growth by 2.54% (right) decline by 0.39%