## Part II: Modeling via systems

- Mathematics of system of differential equations
- Models of interacting populations
- Chemotaxis models in [E-K] (biochemistry)
- Models of infectious diseases
- Models of neural impulses

**Modeling with system of differential equations**

No species lives in isolation.

The interactions among species give interesting models.

**Model 1a**: (Lokta-Volterra) Predator-Prey model

Interaction: predator eats prey

Example: fox and rabbit

**Assumptions**:

- If no foxes are present, the rabbits reproduce by the Malthus model;
- Without rabbits to eat, the fox population declines at a rate proportional to its size;
- The rate at which the rabbits are eaten is proportional to the rate at which the foxes and rabbits interact;
- The rate at which the foxes are born is proportional to the rate at which the rabbits are eaten by the foxes.

**Variables and parameters**:

$t$-time, $R(t)$-population of rabbits, $F(t)$-population of foxes

$a$: growth rate per capita of the rabbits
$b$: death rate per capita of the foxes
$c$: constant of proportionality that measures the number of rabbits eaten and the interaction between rabbits and foxes
$d$: constant of proportionality that measures the number of foxes born and the interaction between rabbits and foxes

$$\frac{dR}{dt} = aR - cFR$$
$$\frac{dF}{dt} = -bF + dFR$$

**History**:

$$\frac{dR}{dt} = aR - cFR$$
$$\frac{dF}{dt} = -bF + dFR$$

Canadian lynx and snowshoe hare (observed in 1840)

Lokta-Volterra predator-prey model (1925-1926)

Alfred James Lotka (1925)

Vito Volterra (1926):
oscillations in fish population in the Mediterranean

**Model 1b**: logistic prey population

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{N}\right) - cFR$$
$$\frac{dF}{dt} = -bF + dFR$$

**Model 1c**: Holling type predation

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{N}\right) - \frac{cFR}{1 + eF}$$
$$\frac{dF}{dt} = -bF + \frac{dFR}{1 + eF}$$

**First order system of ODEs**:

$$\frac{dR}{dt} = 0.1R - 0.2FR$$
$$\frac{dF}{dt} = -0.2F + 0.04FR$$

Equilibrium solutions: $0.1R - 0.2FR = 0$, $-0.2F + 0.04FR = 0$
$(R, F) = (0, 0)$, $(R, F) = (5, 0.5)$

Analytic method: possible but hard

Qualitative method: more efficient

Numerical method: not covered in class, but we will use `pplane`

**Qualitative tools:**

$$\frac{dR}{dt} = 0.1R - 0.2FR$$
$$\frac{dF}{dt} = -0.2F + 0.04FR$$
$$R(0) = R_0 > 0, \quad F(0) = F_0 > 0.$$

a solution: $R(t)$ and $F(t)$.

Graphing program: `pplane`

**Two kinds of graphs**:

solution orbits: $(R(t), F(t))$ on $(R, F)$ plane

solution curves: $(t, R(t))$, $(t, F(t))$ on $(t, R - F)$ plane

## Orbit Graph: phase portrait

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

$(x, y)$-plane is called **phase plane**,

For fixed $t$, $(x(t), y(t))$ is a point on the phase plane

The solution $(x(t), y(t))$ is a moving point (an orbit) on the phase plane

The slope mark at $(x, y)$ is the vector $(f(x, y), g(x, y))$

The slope is $\dfrac{g(x,y)}{f(x,y)}$ since $\dfrac{dy}{dx} = \dfrac{g(x,y)}{f(x,y)}$.

**Vector form of system of equations**:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

**Vector** is a pair of numbers, and we can use a line segment with direction to represent it.

The length of a vector $(x, y)$ is $\sqrt{x^2 + y^2}$.

The right hand side of the equation is a **vector field**.

**Different kinds of functions**:

$P(t)$: function (one variable, one function)

$P(x,y)$: multi-variable function (two variables, one function)

$(P(t),Q(t))$: vector valued function (one variable, two functions)

$(P(x,y),Q(x,y))$: vector field (two variables, two functions)

**Comparison of Behavior of the solutions in 1-d and 2-d** :

| | 1-d | 2-d |
|---|---|---|
| solution | $x(t)$ | $(x(t), y(t))$ |
| solution curves | on $t - x$ graph | on $t - (x, y)$ graph |
| monotonicity | increasing or decreasing | may not be monotone |
| orbit | a moving point<br>on phase line | a moving point<br>on phase plane |
| asymptotic<br>behavior<br>$t \to \infty$ | (i) tends to equilibrium<br>(ii) goes away to infinity | (i) tends to equilibrium<br>(ii) goes away to infinity<br>(iii) tends to a periodic orb |

Note: there is no chaos in 1-d or 2-d, but in 3-d system, chaos is another asymptotic behavior

## Studies of phase portrait (1): nullclines

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

The sets $f(x, y) = 0$ and $g(x, y) = 0$ are curves on the phase portrait, and these curves are called **nullclines**.

$f(x, y) = 0$ is the $x$-nullcline, where the vector field $(f, g)$ is vertical.

$g(x, y) = 0$ is the $y$-nullcline, where the vector field $(f, g)$ is horizontal.

The nullclines divide the phase portraits into regions, and in each region, the direction of vector field must be one of the following:

<u>north-east, south-east, north-west and south-west</u>

(So nullclines are where the vector field is exactly east, west, north and south)

In each region, we use an arrow to indicate the direction.

(In 1-d, we use only up-arrow and down-arrow in phase lines.)

## Studies of phase portrait (2): equilibrium points

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

Equilibrium points are points where $f(x, y) = 0$ and $g(x, y) = 0$.

Equilibrium points are the intersection points of $x$-nullcline and $y$-nullcline.

Equilibrium points are constant solutions of the system.

Equilibrium points are also called steady state solutions, fixed points, etc.

## Qualitative analysis from nullclines:

Suppose that there is a solution from a point in one of the regions formed by nullclines, then there is only three possibilities for the orbit:

**A.** tends to an equilibrium on the border of this region

**B.** goes away to infinity

**C.** enter other neighboring region following the arrow

More information is needed for equilibrium points to further determination.