Part II: Modeling via systems

- Mathematics of system of differential equations
- Models of interacting populations
- Chemotaxis models in [E-K] (biochemistry)
- Models of infectious diseases
- Models of neural impulses

Modeling with system of differential equations

No species lives in isolation.

The interactions among species give interesting models.

Model 1a: (Lokta-Volterra) Predator-Prey model

Interaction: predator eats prey

Example: fox and rabbit

Assumptions:

- If no foxes are present, the rabbits reproduce by the Malthus model;

- Without rabbits to eat, the fox population declines at a rate proportional to its size;

- The rate at which the rabbits are eaten is proportional to the rate at which the foxes and rabbits interact;

- The rate at which the foxes are born is proportional to the rate at which the rabbits are eaten by the foxes.

Variables and parameters:

t-time, $R(t)$-population of rabbits, $F(t)$-population of foxes

a: growth rate per capita of the rabbits
b: death rate per capita of the foxes
c: constant of proportionality that measures the number of rabbits eaten and the interaction between rabbits and foxes
d: constant of proportionality that measures the number of foxes born and the interaction between rabbits and foxes

$$\frac{dR}{dt} = aR - cFR$$
$$\frac{dF}{dt} = -bF + dFR$$

History:

$$\frac{dR}{dt} = aR - cFR$$
$$\frac{dF}{dt} = -bF + dFR$$

Canadian lynx and snowshoe hare (observed in 1840)

Lokta-Volterra predator-prey model (1925-1926)

Alfred James Lotka (1925)

Vito Volterra (1926):
oscillations in fish population in the Mediterranean

Model 1b: logistic prey population

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{N}\right) - cFR$$
$$\frac{dF}{dt} = -bF + dFR$$

Model 1c: Holling type predation

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{N}\right) - \frac{cFR}{1 + eF}$$
$$\frac{dF}{dt} = -bF + \frac{dFR}{1 + eF}$$

First order system of ODEs:

$$\frac{dR}{dt} = 0.1R - 0.2FR$$
$$\frac{dF}{dt} = -0.2F + 0.04FR$$

Equilibrium solutions: $0.1R - 0.2FR = 0$, $-0.2F + 0.04FR = 0$
$(R, F) = (0, 0)$, $(R, F) = (5, 0.5)$

Analytic method: possible but hard

Qualitative method: more efficient

Numerical method: not covered in class, but we will use `pplane`

Qualitative tools:

$$\begin{aligned} &\frac{dR}{dt} = 0.1R - 0.2FR \\ &\frac{dF}{dt} = -0.2F + 0.04FR \\ &R(0) = R_0 > 0, \;\; F(0) = F_0 > 0. \end{aligned}$$

a solution: $R(t)$ and $F(t)$.

Graphing program: `pplane`

Two kinds of graphs:

solution orbits: $(R(t), F(t))$ on (R, F) plane

solution curves: $(t, R(t))$, $(t, F(t))$ on $(t, R-F)$ plane

Orbit Graph: phase portrait

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

(x,y)-plane is called **phase plane**,

For fixed t, $(x(t),y(t))$ is a point on the phase plane

The solution $(x(t),y(t))$ is a moving point (an orbit) on the phase plane

The slope mark at (x,y) is the vector $(f(x,y),g(x,y))$

The slope is $\frac{g(x,y)}{f(x,y)}$ since $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$.

Vector form of system of equations:

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

Vector is a pair of numbers, and we can use a line segment with direction to represent it.

The length of a vector (x, y) is $\sqrt{x^2 + y^2}$.

The right hand side of the equation is a **vector field**.

Different kinds of functions:

$P(t)$: function (one variable, one function)

$P(x, y)$: multi-variable function (two variables, one function)

$(P(t), Q(t))$: vector valued function (one variable, two functions)

$(P(x, y), Q(x, y))$: vector field (two variables, two functions)

Comparison of Behavior of the solutions in 1-d and 2-d :

	1-d	2-d
solution	$x(t)$	$(x(t), y(t))$
solution curves	on $t - x$ graph	on $t - (x, y)$ graph
monotonicity	increasing or decreasing	may not be monotone
orbit	a moving point on phase line	a moving point on phase plane
asymptotic behavior $t \to \infty$	(i) tends to equilibrium (ii) goes away to infinity	(i) tends to equilibrium (ii) goes away to infinity (iii) tends to a periodic orb

Note: there is no chaos in 1-d or 2-d, but in 3-d system, chaos is another asymptotic behavior

Studies of phase portrait (1): nullclines

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

The sets $f(x, y) = 0$ and $g(x, y) = 0$ are curves on the phase portrait, and these curves are called **nullclines**.

$f(x, y) = 0$ is the x-nullcline, where the vector field (f, g) is vertical.

$g(x, y) = 0$ is the y-nullcline, where the vector field (f, g) is horizontal.

The nullclines divide the phase portraits into regions, and in each region, the direction of vector field must be one of the following:

<u>north-east, south-east, north-west and south-west</u>

(So nullclines are where the vector field is exactly east, west, north and south)

In each region, we use an arrow to indicate the direction.

(In 1-d, we use only up-arrow and down-arrow in phase lines.)

Studies of phase portrait (2): equilibrium points

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

Equilibrium points are points where $f(x,y) = 0$ and $g(x,y) = 0$.

Equilibrium points are the intersection points of x-nullcline and y-nullcline.

Equilibrium points are constant solutions of the system.

Equilibrium points are also called steady state solutions, fixed points, etc.

Qualitative analysis from nullclines:

Suppose that there is a solution from a point in one of the regions formed by nullclines, then there is only three possibilities for the orbit:

A. tends to an equilibrium on the border of this region

B. goes away to infinity

C. enter other neighboring region following the arrow

More information is needed for equilibrium points to further determination.