Studies of phase portrait (1): nullclines

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

The sets $f(x, y) = 0$ and $g(x, y) = 0$ are curves on the phase portrait, and these curves are called **nullclines**.

$f(x, y) = 0$ is the x-nullcline, where the vector field (f, g) is vertical.

$g(x, y) = 0$ is the y-nullcline, where the vector field (f, g) is horizontal.

The nullclines divide the phase portraits into regions, and in each region, the direction of vector field must be one of the following:

north-east, south-east, north-west and south-west

(So nullclines are where the vector field is exactly east, west, north and south)

In each region, we use an arrow to indicate the direction.

(In 1-d, we use only up-arrow and down-arrow in phase lines.)

Studies of phase portrait (2): equilibrium points

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

Equilibrium points are points where $f(x, y) = 0$ and $g(x, y) = 0$.

Equilibrium points are the intersection points of x-nullcline and y-nullcline.

Equilibrium points are constant solutions of the system.

Equilibrium points are also called steady state solutions, fixed points, etc.

Qualitative analysis from nullclines:

Suppose that there is a solution from a point in one of the regions formed by nullclines, then there is only three possibilities for the orbit:

A. tends to an equilibrium on the border of this region

B. goes away to infinity

C. enter other neighboring region following the arrow

More information is needed for equilibrium points to further determination.

Studies of phase portrait (3): linearization at equilibrium

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

Suppose that (x_0, y_0) is an equilibrium point. Near it, the behavior of the solutions is governed by the linearized equation

$$\frac{dx}{dt} = \frac{\partial f(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y - y_0)$$
$$\frac{dy}{dt} = \frac{\partial g(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial g(x_0,y_0)}{\partial y}(y - y_0)$$

Since $f(x,y) \approx f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y - y_0)$

and $g(x,y) \approx g(x_0,y_0) + \frac{\partial g(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial g(x_0,y_0)}{\partial y}(y - y_0)$

Quick Review of Multi-variable calculus:

Linearization in 1-d: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

Linearization in 2-d:
$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y - y_0)$$

partial derivative: derivative of f w.r.t. x when y is fixed

Notation: $\dfrac{\partial f(x_0,y_0)}{\partial x}$ or $f_x(x_0,y_0)$

Jacobian: all four partial derivatives of a vector field in a matrix
$$\begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix}$$

Classify the pictures of linear system:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \text{ or } \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are the numbers λ such that $Av = \lambda v$ has solution (λ, v), and v is called eigenvector.

Eigenvalues are solved by equation:

$$(a - \lambda)(d - \lambda) - bc = 0$$

Case 1: two real distinctive eigenvalues
Case 2: two real repeated eigenvalues
Caes 3: two complex eigenvalues

Linearization Theorem in 1-d:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

- if $f'(y_0) < 0$, then y_0 is a sink;
- if $f'(y_0) > 0$, then y_0 is a source;
- if $f'(y_0) = 0$, then y_0 can be any type, but in addition
  - if $f''(y_0) > 0$ or $f''(y_0) < 0$, then y_0 is a node.

Linearization Theorem in 2-d is much more complicated, but the principle is that the role played by $f'(x_0)$ is now played by the eigenvalues of the Jacobian.

A solution is a **stable orbit** if $\mathbf{Y}(t) = (0,0)$ when $t \to \infty$.
A solution is a **unstable orbit** if $\mathbf{Y}(t) = (0,0)$ when $t \to -\infty$.

A. Source

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x + 2y \\ x + 3y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 4$

1. $(0,0)$ is the only equilibrium point, and any non-zero solution is a unstable orbit.

2. There are two straight line solutions on the direction of eigen-vectors.

B. sink

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -2x - 2y \\ -x - 3y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -4$

1. $(0,0)$ is the only equilibrium point, and any non-zero solution is a stable orbit.

2. There are two straight line solutions on the direction of eigen-vectors.

C. saddle

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x + 3y \\ x - y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = -2$

1. $(0, 0)$ is the only equilibrium point.
2. There is one unstable orbit on the direction of eigenvector associated with $\lambda_1 = 2$, and it's a straight line solution.
3. There is one stable orbit on the direction of eigenvector associated with $\lambda_2 = -2$, and it's a straight line solution.
4. Any non-straight-line solution satisfies
(i) $\lim_{t \to \pm\infty} \mathbf{Y}(t) = \infty$
(ii) when $t \to \infty$, the solution tends to the unstable solution,
(iii) when $t \to -\infty$, the solution tends to the stable solution.

D. Spiral sink

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -0.2x - 3y \\ 3x - 0.2y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$

1. $(0, 0)$ is the only equilibrium point, and any non-zero solution is a stable orbit.

2. There is no straight line solutions.

3. Any non-zero solution spiral toward the origin, around the origin infinitely many times.

E. spiral source

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0.2x + 3y \\ -3x + 0.2y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 0.2 + 3i$ and $\lambda_2 = 0.2 - 3i$

1. $(0, 0)$ is the only equilibrium point, and any non-zero solution is an unstable orbit.

2. There is no straight line solutions.

3. Any non-zero solution spiral away from the origin, around the origin infinitely many times.

Classification of linear system:

Two real eigenvalues:

1. $\lambda_1 > \lambda_2 > 0$: source (unstable node in [E-K])
2. $\lambda_1 > \lambda_2 = 0$: degenerate source
3. $\lambda_1 > 0 > \lambda_2$: saddle (same in [E-K])
4. $\lambda_1 = 0 > \lambda_2$: degenerate sink
5. $0 > \lambda_1 > \lambda_2$: sink (stable node in [E-K])

Two complex eigenvalues: $\lambda_\pm = a \pm bi$

1. $a > 0$: spiral source (unstable spiral in [E-K])
2. $a = 0$: center (neutral center in [E-K])
3. $a < 0$: spiral sink (stable spiral in [E-K])

One real eigenvalue: $\lambda_1 = \lambda_2 = \lambda$

1. $\lambda > 0$: star source or "trying to spiral source"
2. $\lambda = 0$: parallel lines
3. $\lambda < 0$: star sink or "trying to spiral sink"

Generic Cases: (most likely, not fragile)

Source (unstable node in [E-K])
Sink (stable node in [E-K])
Saddle (saddle in [E-K])
Spiral source (unstable spiral in [E-K])
Spiral sink (stable spiral in [E-K])

Linearization Theorem in 2-d:

Suppose that (x_0, y_0) is an equilibrium point of $x' = f(x,y)$ and $y' = g(x,y)$, and the eigenvalues of Jacobian $J(x_0, y_0)$ are λ_1 and λ_2.

(1) $\lambda_1 > \lambda_2 > 0$, then the system is a source;
(2) $\lambda_1 > 0 > \lambda_2$, then the system is a saddle;
(3) $0 > \lambda_1 > \lambda_2$, then the system is a sink;
(4) $\lambda_{1,2} = a \pm bi$, $a > 0$, then the system is a spiral source;
(5) $\lambda_{1,2} = a \pm bi$, $a < 0$, then the system is a spiral sink;
(6) If the eigenvalues are other cases, then you need other information to determine the solution behavior near the equilibrium point.