A solution is a **stable orbit** if $\mathbf{Y}(t) = (0,0)$ when $t \to \infty$.
A solution is a **unstable orbit** if $\mathbf{Y}(t) = (0,0)$ when $t \to -\infty$.

**A. Source**

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x + 2y \\ x + 3y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 4$

1. $(0,0)$ is the only equilibrium point, and any non-zero solution is a unstable orbit.

2. There are two straight line solutions on the direction of eigen-vectors.

**B. sink**

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -2x - 2y \\ -x - 3y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -4$

1. $(0, 0)$ is the only equilibrium point, and any non-zero solution is a stable orbit.

2. There are two straight line solutions on the direction of eigen-vectors.

**C. saddle**

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x + 3y \\ x - y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = -2$

1. $(0,0)$ is the only equilibrium point.
2. There is one unstable orbit on the direction of eigenvector associated with $\lambda_1 = 2$, and it's a straight line solution.
3. There is one stable orbit on the direction of eigenvector associated with $\lambda_2 = -2$, and it's a straight line solution.
4. Any non-straight-line solution satisfies

(i) $\lim_{t \to \pm\infty} \mathbf{Y}(t) = \infty$

(ii) when $t \to \infty$, the solution tends to the unstable solution,

(iii) when $t \to -\infty$, the solution tends to the stable solution.

**D. Spiral sink**

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -0.2x - 3y \\ 3x - 0.2y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$

1. $(0, 0)$ is the only equilibrium point, and any non-zero solution is a stable orbit.

2. There is no straight line solutions.

3. Any non-zero solution spiral toward the origin, around the origin infinitely many times.

**E. spiral source**

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0.2x + 3y \\ -3x + 0.2y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 0.2 + 3i$ and $\lambda_2 = 0.2 - 3i$

1. $(0, 0)$ is the only equilibrium point, and any non-zero solution is an unstable orbit.

2. There is no straight line solutions.

3. Any non-zero solution spiral away from the origin, around the origin infinitely many times.

**Classification of linear system:**

Two real eigenvalues:

1. $\lambda_1 > \lambda_2 > 0$: source (unstable node in [E-K])
2. $\lambda_1 > \lambda_2 = 0$: degenerate source
3. $\lambda_1 > 0 > \lambda_2$: saddle (same in [E-K])
4. $\lambda_1 = 0 > \lambda_2$: degenerate sink
5. $0 > \lambda_1 > \lambda_2$: sink (stable node in [E-K])

Two complex eigenvalues: $\lambda_\pm = a \pm bi$

1. $a > 0$: spiral source (unstable spiral in [E-K])
2. $a = 0$: center (neutral center in [E-K])
3. $a < 0$: spiral sink (stable spiral in [E-K])

One real eigenvalue: $\lambda_1 = \lambda_2 = \lambda$

1. $\lambda > 0$: star source or "trying to spiral source"
2. $\lambda = 0$: parallel lines
3. $\lambda < 0$: star sink or "trying to spiral sink"

Generic Cases: (most likely, not fragile)

Source (unstable node in [E-K])
Sink (stable node in [E-K])
Saddle (saddle in [E-K])
Spiral source (unstable spiral in [E-K])
Spiral sink (stable spiral in [E-K])

## Linearization Theorem in 2-d:

Suppose that $(x_0, y_0)$ is an equilibrium point of $x' = f(x, y)$ and $y' = g(x, y)$, and the eigenvalues of Jacobian $J(x_0, y_0)$ are $\lambda_1$ and $\lambda_2$.

(1) $\lambda_1 > \lambda_2 > 0$, then the system is a source;
(2) $\lambda_1 > 0 > \lambda_2$, then the system is a saddle;
(3) $0 > \lambda_1 > \lambda_2$, then the system is a sink;
(4) $\lambda_{1,2} = a \pm bi$, $a > 0$, then the system is a spiral source;
(5) $\lambda_{1,2} = a \pm bi$, $a < 0$, then the system is a spiral sink;
(6) If the eigenvalues are other cases, then you need other information to determine the solution behavior near the equilibrium point.

**Model 1b**: logistic prey population

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{N}\right) - bFR$$
$$\frac{dF}{dt} = -cF + dFR$$

$R$-nullclines: $R = 0$ or $a - \frac{aR}{N} - bF = 0$.

$F$-nullclines: $F = 0$ or $-c + dR = 0$

Equilibrium points: $E_1 = (0,0)$, $E_2 = (N,0)$, $E_3 = (\frac{c}{d}, \frac{a(dN-c)}{bdN})$.

At $E_1 = (0, 0)$, $J = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$, so it is a saddle.

At $E_2 = (N, 0)$, $J = \begin{pmatrix} -a & -cN \\ 0 & dN \end{pmatrix}$, so it is a saddle.

At $E_3 = (\frac{b}{d}, \frac{a(dN-b)}{cdN})$, $J = \begin{pmatrix} -(ac)/(dN) & -(bc)/d \\ a(dN-c)/(bN) & 0 \end{pmatrix}$, so it is a sink or spiral sink.