Malthus Model:

Assumption: the reproduction rate is proportional to the size of the population

$$\frac{dP}{dt} = kP, \;\; k = \text{growth rate per capita}$$

Solution: $P(t) = P(0)e^{kt}$

$k > 0$: exponential growth, $k < 0$: exponential decay

Proposed by Thomas Robert Malthus (1798) (British) :
An essay on the principle of population (available online)

Validity: good for bacteria in an unlimited environment or computer virus, otherwise not reasonable due the the limitation of the resource

Logistic Model:

Assumption: the reproduction rate is proportional to the size of the population when the population size is small, and the growth is negative when the size is large

$$\frac{dP}{dt} = P \cdot g(P), \;\; g(P) = \text{growth rate per capita}$$

Choose the simplest form which fulfils the assumptions:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

$k =$ maximum growth rate per capita, $N =$ carrying capacity

Proposed by Pierre Francois Verhulst (1838) (Belgian)

Examples: bacteria, yeast in a limited environment. (laboratory experiments)

Example 1: A biologist prepares a culture. After 1 day of growth, the biologist counts 1000 cells. After 2 days of growth, he counts 3000. Assuming a Malthus model, what is the reproduction rate and how many cells were present initially?

Qualitative analysis of Logistic equation:

$$\frac{dP}{dt} = P\left(1 - \frac{P}{4}\right)$$

- $P = 0$ and $P = 4$ are equilibrium solutions (constant solutions.)

- When $0 < P(0) < 4$, the solution is increasing, since $\frac{dP}{dt} = P\left(1 - \frac{P}{4}\right) > 0$, and $\lim_{t\to\infty} P(t) = 4$.

- When $P(0) > 4$, the solution is decreasing, since $\frac{dP}{dt} = P\left(1 - \frac{P}{4}\right) < 0$, and $\lim_{t\to\infty} P(t) = 4$.

Analytical solution of Logistic equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right), \quad P(0) = P_0$$

$$P(t) = \frac{NP_0}{(N - P_0)e^{-kt} + P_0}$$

Example 2: A population is observed to obey the Logistic equation with eventual population 20,000. The initial population is 1000, and 8 hours later, the observed population is 1200. Find the reproductive rate and the time required for the population to reach 75% of its carry capacity.

(answer: $k = 0.0241$, $T = 167.76$ hours.)

Method of separation of variables:

$$\frac{dy}{dt} = f(t)g(y)$$

$$\frac{dy}{g(y)} = f(t)dt \Rightarrow \int \frac{dy}{g(y)} = \int f(t)dt$$

Then solve y in term of t if possible.

Example 3:

(1) $y' = ty$, $y(0) = 3$.

(2) $x' = \dfrac{2tx}{1+x}$.