

## Malthus Model:

Assumption: the reproduction rate is proportional to the size of the population

$$\frac{dP}{dt} = kP, \quad k = \text{growth rate per capita}$$

Solution:  $P(t) = P(0)e^{kt}$

$k > 0$ : exponential growth,  $k < 0$ : exponential decay

Proposed by Thomas Robert Malthus (1798) (British) :  
*An essay on the principle of population* (available online)

Validity: good for bacteria in an unlimited environment or computer virus, otherwise not reasonable due to the limitation of the resource

## Logistic Model:

Assumption: the reproduction rate is proportional to the size of the population when the population size is small, and the growth is negative when the size is large

$$\frac{dP}{dt} = P \cdot g(P), \quad g(P) = \text{growth rate per capita}$$

Choose the simplest form which fulfils the assumptions:

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right)$$

$k$  = maximum growth rate per capita,  $N$  = carrying capacity

Proposed by Pierre Francois Verhulst (1838) (Belgian)

Examples: bacteria, yeast in a limited environment. (laboratory experiments)

**Example 1**: A biologist prepares a culture. After 1 day of growth, the biologist counts 1000 cells. After 2 days of growth, he counts 3000. Assuming a Malthus model, what is the reproduction rate and how many cells were present initially?

## Qualitative analysis of Logistic equation:

$$\frac{dP}{dt} = P \left( 1 - \frac{P}{4} \right)$$

- $P = 0$  and  $P = 4$  are equilibrium solutions (constant solutions.)
- When  $0 < P(0) < 4$ , the solution is increasing, since  $\frac{dP}{dt} = P \left( 1 - \frac{P}{4} \right) > 0$ , and  $\lim_{t \rightarrow \infty} P(t) = 4$ .
- When  $P(0) > 4$ , the solution is decreasing, since  $\frac{dP}{dt} = P \left( 1 - \frac{P}{4} \right) < 0$ , and  $\lim_{t \rightarrow \infty} P(t) = 4$ .

## Analytical solution of Logistic equation:

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right), \quad P(0) = P_0$$

$$P(t) = \frac{NP_0}{(N - P_0)e^{-kt} + P_0}$$

**Example 2:** A population is observed to obey the Logistic equation with eventual population 20,000. The initial population is 1000, and 8 hours later, the observed population is 1200. Find the reproductive rate and the time required for the population to reach 75% of its carry capacity.

(answer:  $k = 0.0241$ ,  $T = 167.76$  hours.)

## Method of separation of variables:

$$\frac{dy}{dt} = f(t)g(y)$$

$$\frac{dy}{g(y)} = f(t)dt \Rightarrow \int \frac{dy}{g(y)} = \int f(t)dt$$

Then solve  $y$  in term of  $t$  if possible.

### Example 3:

(1)  $y' = ty, y(0) = 3.$

(2)  $x' = \frac{2tx}{1+x}.$