

**Direction field:** (slope field)

$$\frac{dP}{dt} = f(t, P)$$

If the function  $P(t)$  is a solution of the equation and if its graph passes through the point  $(t_0, P_0)$  where  $P_0 = y(t_0)$ , then the differential equation says that the derivative  $dP/dt$  at  $t = t_0$  is given by the number  $f(t_0, P_0)$ .

Direction field: at each point selected, we draw a minitangent line (we call it slope mark) whose slope is  $f(t, P)$ .

## Autonomous Equation:

$$\frac{dP}{dt} = f(P)$$

The direction field of an autonomous equation does not depend on  $t$ , so the slope marks at  $(t_1, P)$  and  $(t_2, P)$  are same. All solutions are parallel in  $P$  direction in the sense that if  $P(t)$  is a solution, then  $P(t + c)$  is also a solution. So we only need to draw slope marks for a fixed  $t$ .

## Phase line:

Phase line is a simplified direction field for autonomous equation only. On a vertical line, we indicate the positive derivative by an arrow pointing up, and negative derivative by an arrow pointing down.

## Equilibrium solutions:

$$\frac{dP}{dt} = f(P)$$

If for some number  $a$ ,  $f(a) = 0$ , then  $P(t) = a$  is an equilibrium solution (constant solution).

## How to draw the phase line:

- (1) Draw the  $P$ -line. (usually vertical)
- (2) Find and mark the equilibrium points.
- (3) Find the intervals of  $P$ -values for which  $f(P) > 0$ , and draw arrows pointing up in these intervals.
- (4) Find the intervals of  $P$ -values for which  $f(P) < 0$ , and draw arrows pointing down in these intervals.

## Euler's Method:

$$\frac{dP}{dt} = f(t, P), \quad P(t_0) = P_0$$

Idea: use tangent line to approximate the solution curve

$$P - P_0 = P'(t_0)(t - t_0) = f(t_0, P_0)(t - t_0)$$

Approximate the value at  $t = t_1$ :

$$P_1 = P(t_1) = P_0 + f(t_0, P_0)(t_1 - t_0)$$

General formula:

$$t_n = t_{n-1} + \Delta t, \quad P_n = P_{n-1} + f(t_{n-1}, P_{n-1})\Delta t$$

$\Delta t$  is the step size

## Model 3a: Allee effect:

### Assumptions:

- (1) If the population is too large, then the growth is negative.
- (2a) If the population is too small, then the growth is negative.

Example: The fox squirrel (a small mammal native to the Rocky Mountains.) If the population is too small, fertile adults run the risk of not being able to find suitable mates, so the growth is negative.

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) \left(\frac{P}{M} - 1\right)$$

$k$  = growth rate per capita coefficient,  $N$  = carrying capacity,  $M$  = sparsity constant, and  $0 < M < N$ .

## Model 3b: Allee effect:

### Assumptions:

- (1) If the population is too large, then the growth is negative.  
(2a) If the population is too small, then the growth rate per capita is positive but not the largest.

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) \left(\frac{P}{M} - 1\right)$$

$k$  = growth rate per capita coefficient,  $N$  = carrying capacity,  
 $M$  = no meaning here, and  $M < 0 < -M < N$ .

An alternative form:

$$\frac{dP}{dt} = kP[1 - (P - b)^2]$$

$k$  = maximum growth rate per capita,  $b$  = the population size when the growth rate per capita is the largest.  $b > 0$

Case a:  $b > 1$

Case b:  $0 < b < 1$