

## Model 3a: Allee effect:

### Assumptions:

- (1) If the population is too large, then the growth is negative.
- (2a) If the population is too small, then the growth is negative.

Example: The fox squirrel (a small mammal native to the Rocky Mountains.) If the population is too small, fertile adults run the risk of not being able to find suitable mates, so the growth is negative.

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) \left(\frac{P}{M} - 1\right)$$

$k$  = growth rate per capita coefficient,  $N$  = carrying capacity,  $M$  = sparsity constant, and  $0 < M < N$ .

## Model 3b: Allee effect:

### Assumptions:

- (1) If the population is too large, then the growth is negative.  
(2b) If the population is too small, then the growth rate per capita is positive but not the largest.

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) \left(\frac{P}{M} - 1\right)$$

$k$  = growth rate per capita coefficient,  $N$  = carrying capacity,  
 $M$  = no meaning here, and  $M < 0 < -M < N$ .

An alternative form:

$$\frac{dP}{dt} = kP[1 - c(P - b)^2]$$

$k$  = maximum growth rate per capita,  $b$  = the population size when the growth rate per capita is the largest.  $b > 0$

Case a:  $b^2c > 1$

Case b:  $0 < b^2c < 1$

**Example 1**: A biologist start with 10 cells in a culture. After each of the first four days, he counts 25, 62, 154, and 375 cells. Assuming a Malthus model, what is the reproduction rate? What will be the number of cells after 10 days?

**Linear Regression**: The goal of linear regression is to adjust the values of slope and intercept to find the line that best predicts Y from X. More precisely, the goal of regression is to minimize the sum of the squares of the vertical distances of the points from the line.

**Method of linear regression**:

- (a) get a set of data:  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ .
- (b) put it into a computer linear regression program

## Mechanism of linear regression:

Data:  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ .

Goal: Find a line  $y = kx + b$  minimize the sum of the squares of the vertical distances of the points from the line.

Vertical distance from  $(x_i, y_i)$  to  $y = kx + b$ :  $kx_i + b - y_i$

Mathematical problem: Find  $k$  and  $b$  which minimize

$$f(k, b) = \sum_{i=1}^n (kx_i + b - y_i)^2$$

$$\frac{\partial f}{\partial k} = \sum_{i=1}^n 2(kx_i + b - y_i)x_i = 0, \quad \frac{\partial f}{\partial b} = \sum_{i=1}^n 2(kx_i + b - y_i) = 0.$$

So solve  $k$  and  $b$  from

$$\left( \sum_{i=1}^n x_i \right) k + nb = \sum_{i=1}^n y_i \text{ and}$$
$$\left( \sum_{i=1}^n x_i^2 \right) k + \left( \sum_{i=1}^n x_i \right) b = \sum_{i=1}^n x_i y_i.$$

**Data fitting for Malthus model:** (estimate  $k$  and  $P_0$ )

(a) Get a set of data:  $(t_1, P_1), (t_2, P_2), \dots, (t_n, P_n)$ .

(b) Take  $\ln$  to  $P_i$ , let  $Q_i = \ln(P_i)$ , and get new data set  $(t_1, Q_1), (t_2, Q_2), \dots, (t_n, Q_n)$ .

(c) Put the data set  $(t_i, Q_i)$  to your linear regression program and get the output slope  $k$  and intercept  $b$ .

(d) In the solution of Malthus model:  $P(t) = P(0)e^{kt}$ , we have  $\ln P(t) = \ln P(0) + kt$ , so  $b = \ln P(0)$ . Then  $P(0) = e^b$ . So an exponential function which best fits the data is

$$P(t) = e^{b+kt}, \text{ where } k \text{ and } b \text{ are found in linear regression}$$

**Data fitting for Logistic model:** (estimate  $k$  and  $N$ ,  $P_0$  is known)

Notice that  $\ln \frac{P}{N - P} = \ln \frac{P_0}{N - P_0} + kt$

(a) Get a set of data:  $(t_1, P_1), (t_2, P_2), \dots, (t_n, P_n)$ .

(b) Let  $Q_i = \ln(P_i) - \ln(N - P_i)$ , and get new data set  $(t_1, Q_1), (t_2, Q_2), \dots, (t_n, Q_n)$ .

(c) Put the data set  $(t_i, Q_i)$  to your linear regression program and get the output slope  $k$  and intercept  $b$ .

(d)  $k$  is the maximum growth rate per capita in the Logistic equation.  $b = \ln \frac{P_0}{N - P_0}$ , from which we can solve  $N$  provided  $P_0$  is known.

## Population Model for human being:

$$\frac{dP}{dt} = [B(t) - D(t)]P \left[ 1 - \frac{P}{N(t)} \right] + I(t) - E(t)$$

$B(t)$ : birth rate per capita at  $t$

$D(t)$ : death rate per capita at  $t$

$N(t)$ : carry capacity due to technology

$I(t)$ : rate of immigration

$E(t)$ : rate of emigration