Stability of an equilibrium point:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

y_0 is a **sink** if any solution with initial condition close to y_0 tends toward y_0 as t increase.

y_0 is a **source** if any solution with initial condition close to y_0 tends toward y_0 as t decrease.

y_0 is a **node** if it is neither a sink nor a source.

Linearization Theorem:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

- if $f'(y_0) < 0$, then y_0 is a sink;

- if $f'(y_0) > 0$, then y_0 is a source;

- if $f'(y_0) = 0$, then y_0 can be any type, but in addition

  - if $f''(y_0) > 0$ or $f''(y_0) < 0$, then y_0 is a node.

Bifurcation: Suppose that the differential equation depends on a parameter. Then we say that a bifurcation occurs if there is a qualitative change in the behavior of solutions as the parameter changes.

Example 1: $\dfrac{dy}{dt} = ky(1-y)$ (no bifurcation)

Example 2: $\dfrac{dy}{dt} = y^2 - \mu$ (saddle-node bifurcation, supercritical)

Example 3: $\dfrac{dy}{dt} = y^3 + \mu y$ (pitchfork bifurcation, subcritical)

Example 4: $\dfrac{dy}{dt} = y^2 - \mu y$ (transcritical bifurcation)

Example 5: Constant yield harvesting

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - h$$

Mathematical Analysis:

1. Nondimensionalization: $u = \dfrac{P}{N}, s = kt,$

$$\frac{du}{dt} = u(1-u) - H, \quad H = \frac{h}{kN}$$

2. Bifurcation: a subcritical saddle-node bifurcation occurs at $H = 0.25$, or $h = 0.25kN$.

3. Qualitative analysis: when $0 < H < 0.25$, $H = 0.25$ and $H > 0.25$.

4. Analytic method: solve the equation? (see homework)

Example 5 (Cont.): Constant yield harvesting

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - h$$

Biological interpretation:

1. When $0 < H < 0.25$, there are two equilibrium points $P_1 > P_2 > 0$; for $P(0) > P_2$, $\lim_{t\to\infty} P(t) = P_1$ and for $0 < P(0) < P_2$, $P(t) < 0$ for $t > t_0$; P_1 is smaller than N, that means the carrying capacity decreases because of harvesting; the behavior of solutions with $P(0) > P_2$ is similar to that of logistic equation; if the initial population is less than P_2, then the population becomes extinct in finite time.

2. When $H > 0.25$, there is no equilibrium points, and for any initial population, the population becomes extinct in finite time.

3. $H = 0.25$ or $h = 0.25kN$ is called Maximum Sustainable Yield (MSY).

Example 6: Constant effort harvesting

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - hP$$

Mathematical Analysis:

1. Nondimensionalization: $u = \dfrac{P}{N}, s = kt,$

$$\frac{du}{dt} = u(1-u) - Hu, \quad H = \frac{h}{kN}$$

2. Bifurcation: no bifurcation if we assume that $0 < H < 1$

3. A different question: for which H, we can get the maximum yield Hu?