

Summary of Part 1:

Population models for single species:

Model 1: Malthus model

Model 2: Logistic model

Model 3a,b: Allee effect

Model 4a,b,c,d,e: Harvesting model

Model 5: Spruce budworm model

Fishing model: management of renewable natural resources
(fish, forest...)

A General Population Model: (deterministic model)

$$\frac{dP}{dt} = [B(t) - D(t)]P \left[1 - \frac{P}{N(t)} \right] + I(t) - E(t)$$

$B(t)$: birth rate per capita at t

$D(t)$: death rate per capita at t

$N(t)$: carry capacity due to technology

$I(t)$: rate of immigration

$E(t)$: rate of emigration

Stochastic model: $P(t)$ not a function, but a probability distribution

Mathematical methods for single ODE:

1. Analytic methods: separation of variables.
2. Numerical method: Euler's method, data fitting.
3. Qualitative methods: direction field, graph of solutions, phase line, equilibrium points and their stability, bifurcation.
4. Nondimensionalization.

Model 4d: Holling's type II model

$$\frac{dQ}{ds} = Q(1 - Q) - \frac{hQ}{1 + aQ}$$

A. when $a = 2$, $Q(1 - Q) - \frac{hQ}{1 + 2Q} = 0$

$$Q = 0 \text{ or } 2Q^2 - Q + (h - 1) = 0, \quad Q = \frac{1 \pm \sqrt{9 - 8h}}{4}$$

Bifurcation diagram: $h = -2Q^2 + Q + 1$ (but in a $h - Q$ graph)

Two bifurcation points:

$h = 9/8$: subcritical saddle-node bifurcation

$h = 1$: transcritical bifurcation

Biological implications when $a = 2$:

1. When $0 < h < 1$, there are two equilibrium points, 0 and $Q_+ = 0.25(1 + \sqrt{9 - 8h})$. The system is similar to logistic equation.
2. When $1 < h < 9/8$, there are three equilibrium points, 0 and $Q_{\pm} = 0.25(1 + \sqrt{9 - 8h})$. The system is similar to logistic equation with allee effect.
3. When $h > 9/8$, 0 is the only equilibrium point. The population will become extinct no matter how large the initial population is.
4. Depending on the value of h , the state of the system is survival, partial survival and extinction.

B. General (a, h) . $Q(1 - Q) - \frac{hQ}{1 + aQ} = 0$

$Q = 0$ or $aQ^2 + (1 - a)Q + (h - 1) = 0$,

$Q_{\pm} = \frac{a - 1 \pm \sqrt{(a + 1)^2 - 4ah}}{2a}$, Basic border line: $h = \frac{(a + 1)^2}{4a}$

when $0 < h < \frac{(a + 1)^2}{4a}$, three equilibrium points

when $h = \frac{(a + 1)^2}{4a}$, two equilibrium points (except $a = 1$)

when $h > \frac{(a + 1)^2}{4a}$, one equilibrium points

But we also count the negative equilibrium points

Delicate but simple analysis:

$$\text{when } Q_+ < 0: Q_+ = \frac{a - 1 + \sqrt{(a + 1)^2 - 4ah}}{2a} < 0$$

$$\text{when } Q_+ \geq 0 \text{ but } Q_- < 0: Q_- = \frac{a - 1 - \sqrt{(a + 1)^2 - 4ah}}{2a} < 0$$

Derivative method:

$$\text{Solve } f(Q) = aQ^2 + (1 - a)Q + (h - 1) = 0$$

$$\text{and } f'(Q) = 2aQ + (1 - a) = 0.$$