

Math 410(510) Notes (2)

Analyzing Single Species Population Models

Ordinary Differential Equations

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Since we learn the word *equation*, we always want to *solve* an equation, which means to find the solution(s) of the equation. For example

$$(1) \quad x^2 - 3x + 2 = 0. \quad \text{Solve: } x_1 = 2, x_2 = 1.$$

Once an equation is solved, that is the end of the story: we find all and only solutions to the equation. But this is not always the case. Another example:

$$(2) \quad x^7 - 5x + 2 = 0.$$

There is no good way to solve the equation. But still we can have ways to get the information of the solutions to the equation. A) By using computer or calculator, we can graph the function $f(x) = x^7 - 5x + 2$, and on the graph we can “see” and roughly locate the roots; B) we can use the Newton’s method or other numerical method to find highly accurate approximate solutions. So for the algebraic equation like (2), first we try to solve it, that is called *analytic method*; if that wouldn’t work, we can analyze its graph, that is called *qualitative method* or *geometric method*; finally we can use *numerical method* to find approximate solution.

For differential equation, all three methods are still useful but techniques are quite different. In this notes, we describe all three methods for equation:

$$(3) \quad \frac{dP}{dt} = f(P).$$

Equation (3) is called a *first order autonomous ordinary differential equation*. First there are two kinds of differential equations: ordinary differential equation (ODE) and partial differential equation (PDE). A solution of an ODE is a function $P(t)$, which depends on only one variable; a solution of a PDE is a function like $P(t, x, y)$, which depends on more than one variables. (3) is called first order since there is only the first order derivative dP/dt in the equation. Here are some higher order equations:

$$(4) \quad P'' + 3P' - 200P = 2; \text{ (2nd order)} \quad \frac{d^3P}{dt^3} = \sin t. \text{ (3rd order)}$$

Finally it is called autonomous since the right hand side of the equation only depends on P but not t . A non-autonomous equation is like: $dP/dt = P^2 + t$, or more general

$$(5) \quad \frac{dP}{dt} = f(t, P).$$

1. Analytic method: how to solve the equation?

This is how it basically goes:

$$\frac{dP}{dt} = f(P) \Rightarrow \frac{dP}{f(P)} = dt \Rightarrow \int \frac{dP}{f(P)} = \int dt \Rightarrow \int \frac{dP}{f(P)} = t + C,$$

where C is a constant, and you should solve the integral, then solve P in term of t . We show that we can do this for the exponential model and logistic model.

Exponential Model: $\frac{dP}{dt} = kP$, $\frac{dP}{P} = kdt$, $\int \frac{dP}{P} = \int kdt$, $\ln|P| = kt + C$, $|P| = e^{kt+C} = Ce^{kt}$, so we obtain $P(t) = Ce^{kt}$, where C is a constant. Note that C here can be any constant, and it changes in e^{kt+C} to Ce^{kt} . Indeed by the property of exponential function we have $e^{kt+C} = e^{kt}e^C$, so our new constant C is e^C in the previous expression. But e^C is still a constant, we don't need to keep track how the constant changes here, so we just still call it C . Also we can drop the absolute value $|\cdot|$ on P if we allow C to be negative, since $|P| = \pm P$.

Initial Value Problem: Since C can be any number in the above expression, then the differential equation $P' = kP$ has *infinite many* solutions. Which one is useful for us? Usually in applications, we know the initial population, *i.e.* $P(0)$, or just population at any given time. Then we can actually solve the constant C by using this information. For example, consider

$$(6) \quad \frac{dP}{dt} = 3P, \quad P(0) = 100.$$

From above, we first get $P(t) = Ce^{3t}$, this is called the general solution of the equation $P' = 3P$. Now we substitute $P(0) = 100$ into the general solution: $100 = P(0) = Ce^0 = C$, so $C = 100$ and the unique solution to the initial value problem is $P(t) = 100e^{3t}$.

In general, the solution of

$$(7) \quad \frac{dP}{dt} = kP, \quad P(0) = P_0$$

is $P(t) = P_0e^{kt}$. When $k > 0$, this population has an exponential growth; when $k < 0$, the population has an exponential decay.

Logistic Equation:

Consider

$$(8) \quad \frac{dP}{dt} = 2P(1 - P), \quad P(0) = 0.6.$$

$\frac{dP}{P(1-P)} = 2dt$, $\left(\frac{1}{P} + \frac{1}{1-P}\right) dP = 2dt$, $\ln|P| - \ln|1-P| = t + C$, $\ln\left|\frac{P}{1-P}\right| = t + C$, $\frac{P}{1-P} = Ce^t$. Since $P(0) = 0.6$, then $\frac{0.6}{0.4} = Ce^0$, so $C = 3/2$. $\frac{P}{1-P} = \frac{3}{2}e^t$, $P = \frac{3}{2}(1-P)e^t$, $P = \frac{3}{2}e^t - \frac{3}{2}e^tP$, so $P(t) = \frac{(3/2)e^t}{1 + (3/2)e^t} = \frac{3e^t}{2 + 3e^t}$.

Why $\frac{dP}{P(1-P)}$ can be broken into $\left(\frac{1}{P} + \frac{1}{1-P}\right) dP$? This is because $\frac{dP}{P(1-P)} = \frac{1}{P} + \frac{1}{1-P}$ by the partial fraction technique, which you should learn in Calculus II (Math 112) course. In general, the solution of

$$(9) \quad \frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right), \quad P(0) = P_0,$$

is $P(t) = \frac{NP_0}{P_0 + (N - P_0)e^{-kt}}$. (see Polking's textbook Chapter 3, page 129-130.)

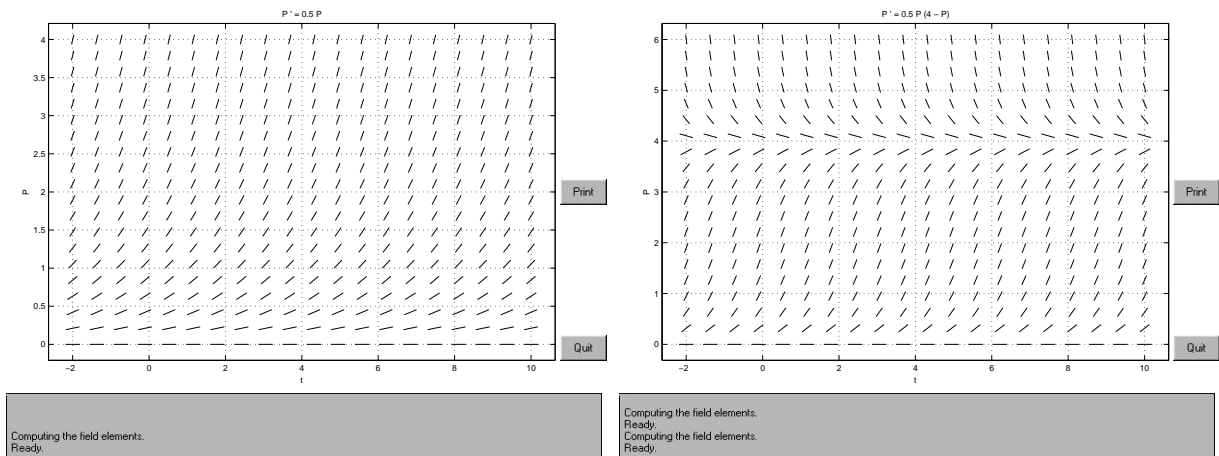
2. Geometric method: how to see the equation?

As you may expect, we cannot solve all the differential equations. In the computer age, we can easily draw various kinds of "graphs" of the differential equations. Here we consider an equation $dP/dt = f(t, P)$. If the function $P(t)$ is a solution of the equation and if its graph passes through the point (t_0, P_0) where

$P_0 = y(t_0)$, then the differential equation says that the derivative dP/dt at $t = t_0$ is given by the number $f(t_0, P_0)$. Geometrically, this means the slope of the tangent line to the graph of $P(t)$ at (t_0, P_0) is $f(t_0, P_0)$. And this must hold for all points on the graph of $P(t)$. In other words, the values of the right-hand side of the differential equation yield the slopes of the tangents at all points on the graph of $P(t)$.

Since we are *given* the function $f(t, P)$ in the differential equation, we obtain a rough idea of the graphs of the solutions to the differential equations by sketching the corresponding **direction field**. We make this sketch by selecting points in the $t - P$ plane and computing the numbers $f(t, P)$ at these points. At each point selected, we draw a minitangent line (we call it slope mark) whose slope is $f(t, P)$. Once we have a lot of such slope marks, we can visualize the graphs of the solutions.

Of course this would be a tedious work if you try to do this by hand drawing even if you just select 100 points. A computer program can do this in a perfect way in just one second. Here are the direction fields of exponential and logistic models:



We introduce a few ways to produce the graphs of direction fields on a computer. If you just want to visualize the equation by seeing the direction field, you can use the the following online java applets which draw direction field:

<http://math.rice.edu/~dfield/> (By John Polking from Rice University)

<http://amath.colorado.edu/faculty/sherod/classes/Phasor/phase.html>

<http://www.math.psu.edu/melvin/phase/newphase.html>

While the online softwares are convenient to use, they usually do not have good printing results, and sometime are even not possible to be printed. For good printout, you can go to any on-campus computer labs which have Windows-2000 network computers. Details will be given in another handout.

3. Numerical method: how to compute the approximate solution?

Every budding dynamicist or math biologist should master a third tool: numerical methods. In the old days, numerical methods were impractical because they required enormous amounts of tedious hand-calculations. But all that has changed thanks to the computer. Computers enable us to approximate the solutions to analytically intractable problems, and also to visualize those solutions.

The problem can be posed this way: given the differential equation $P' = f(t, P)$ subject to the condition $P = P_0$ at $t = t_0$, find a way to approximate the solution $P(t)$. We can use the direction field introduced above to help us thinking. We could think the solution $P(t)$ flowing steadily on the P -axis, with velocity $f(t, P)$ at the location P and time t . Imagine we are riding along with a point being carried downstream by the direction field. Initially we are at P_0 when the time is t_0 , and the local velocity is $f(t_0, P_0)$. If we

flow for a short time Δt , we will have moved a distance $f(t_0, P_0)\Delta t$ because distance = rate \times time. Of course, that's not quite right, because our velocity was changing a little bit throughout the step. But over a sufficient *small* step, the velocity will be nearly constant and our approximation should be reasonably good. Hence our new position $P(t_0 + \Delta t)$ is approximately $P_0 + f(t_0, P_0)\Delta t$. Let's call this approximation P_1 . Thus

$$(10) \quad P(t_0 + \Delta t) \approx P_1 = P_0 + f(t_0, P_0)\Delta t.$$

Now we repeat this procedure (we call it *iterations* in mathematics.) We step forward from $P_1 = P_0 + f(t_0, P_0)\Delta t$, to $P_2 = P_1 + f(t_1, P_1)\Delta t$ where $t_1 = t_0 + \Delta t$. In general the update rule is

$$(11) \quad t_n = t_{n-1} + \Delta t, \quad P_n = P_{n-1} + f(t_{n-1}, P_{n-1})\Delta t.$$

This is the simplest possible numerical scheme of finding approximate solution of differential equation, which we call *Euler's method*. Some examples can be found at

<http://www.sosmath.com/diffeq/first/numerical/numerical.html>.

The approximate solution is not a real solution, but it gives a set of data:

$$(t_0, P_0), (t_1, P_1), (t_2, P_2), \dots, (t_n, P_n), \dots$$

which is close to the values of a real solution at t_i . The *error* of the approximate solution and real solution can be measured by $E = |P(t_n) - P_n|$ for a given stepsize Δ . Note that although we may not know the value of $P(t_n)$, but usually we can have an estimate of E . Of course, $E \rightarrow 0$ as $\Delta t \rightarrow 0$, but in some methods, $E \rightarrow 0$ faster than others. For Euler's method, the error is proportional to Δt , so we call it first order method. Next we introduce without explanation two better numerical algorithms. First one is the *improved Euler method*:

$$(12) \quad t_n = t_{n-1} + \Delta t, \quad \widetilde{P}_n = P_{n-1} + f(t_{n-1}, P_{n-1})\Delta t, \quad P_n = P_{n-1} + \frac{1}{2}[f(t_{n-1}, P_{n-1}) + f(t_n, \widetilde{P}_n)]\Delta t,$$

for which the error is proportional to $(\Delta t)^2$, so it is a second order method. In practice, the following *fourth-order Runge-Kutta method* is widely used by scientists:

$$(13) \quad \begin{aligned} \tilde{t} &= t_{n-1} + \frac{1}{2}\Delta t, \\ k_1 &= f(t_{n-1}, P_{n-1})\Delta t, \quad k_2 = f(\tilde{t}, P_{n-1} + \frac{1}{2}k_1)\Delta t, \\ k_3 &= f(\tilde{t}, P_{n-1} + \frac{1}{2}k_2)\Delta t, \quad k_4 = f(t_n, P_{n-1} + k_3)\Delta t, \\ P_n &= P_{n-1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

This method generally gives accurate results without requiring an excessively small stepsize Δ .

All these methods are easy to write program (for example, **C**, **C++**, **Pascal**, **Visual Basic** and probably best today: **Java**) on a computer. And even better, you can do it real time online without knowing any of these programming languages. Here is a link for Euler's method,

<http://mss.math.vanderbilt.edu/~pscroeke/MSS/euler.html>.

I couldn't find a link for the other two methods. But John Polking also write a few routines of Matlab programs for Euler method, 2nd order Runge-Kutta and 4th order Runge-Kutta methods. Details will be given in another handout.