Math 410(510) Notes (3)
Nondimensionalization

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The form of a solution of a differential equation can depend critically on the units one chooses for the various quantities involved. Frequently these choices can lead to substantial problems when numerical approximation techniques such as Euler's method are applied. These difficulties can be controlled or avoided by proper scaling. We describe a technique that changes variables so that the new variables are "dimensionless". This technique will lead to a simple form of the equation with fewer parameters. It makes clear that the parameters can interact in the equation and a simpler combined parameter can suffice for more than one parameter. We illustrate this technique which is called Nondimensionalization, with an example.

Example. Consider the following model of an outbreak of the spruce budworm (see Notes 1 for details on this model):

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - \frac{BP^2}{A^2 + P^2}, \quad P(0) = P_0. \tag{1}$$

We give a step by step approach to nondimensionalize this initial value problem.

Step 1. List all of the variables, parameters, and their dimensions. For the dimensions we use τ for time, and ρ for population in number of worms.

Variable	Dimension	Parameter	Dimension
t	τ	k	$1/\tau$
P	ρ	N	ρ
		A	ρ
		B	ρ/τ
		P_0	ρ

Step 2. Take each variable and create a new variable by dividing by the combination of parameters that has the same dimension in order to create a dimensionless variable. Not that there is not always a unique way to do that, so some experimentation may be necessary. Here we create

$$u = \frac{P}{A}, \quad s = \frac{Bt}{A}.$$

We can use our table of dimensions above to check that these new variables are now dimensionless.

Step 3. Now use the chain rule to derive a new differential equation.

$$\frac{dP}{dt} = \frac{dP}{du} \cdot \frac{du}{ds} \cdot \frac{ds}{dt} = A\frac{du}{ds} \cdot \frac{B}{A} = B\frac{du}{ds}$$

The term $kP\left(1 - \frac{P}{N}\right)$ becomes $kAu\left(1 - \frac{Au}{N}\right)$ and the term $\frac{BP^2}{A^2 + P^2}$ simplifies to $\frac{Bu^2}{1 + u^2}$. Thus the equation simplifies to

$$B\frac{du}{ds} = kAu\left(1 - \frac{Au}{N}\right) - \frac{Bu^2}{1 + u^2}$$

Dividing by B gives

$$\frac{du}{ds} = \frac{kA}{B} u \left(1 - \frac{u}{N/A}\right) - \frac{u^2}{1+u^2}$$

Noting that the combinations of the parameters that occur above leads us to introduce two new dimensionless parameters

$$\alpha = \frac{kA}{B}, \quad \beta = \frac{N}{A}$$

The equation then becomes

$$\frac{du}{ds} = \alpha u \left(1 - \frac{u}{\beta}\right) - \frac{u^2}{1+u^2}$$

At the meantime, the initial condition $P(0) = P_0$ becomes $u(0) = P_0/A$ through the change of variable $u = P/A$. Thus if we introduce another new parameter $\gamma = P_0/A$, then the initial condition becomes

$$u(0) = \gamma$$

Note that the equation has two dimensionless variables s, u and three dimensionless parameters α, β, γ which are combinations of the original parameters. This simplified form of the equation has reduced the number of parameters from 5 to 3, which makes the analysis of the equation simpler.

Exercises

1. Consider the equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - \frac{BP^2}{A^2+P^2}.$$

   Use the following change of variables:

$$(a)\ \ Q = \frac{P}{N}, s = kt, \qquad (b)\ \ Q = \frac{kP}{B}, s = kt,$$

   to get nondimensionalized equations.

2. Consider a population model:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{N}\right) - M,\ x(0) = x_0,$$

   where k, M, N, x_0 are positive parameters.

   (a) In the following table, fill in the dimensions of all parameters in terms of the dimensions of variables.

Variable	Dimension	Parameter	Dimension
t	τ	k	
x	λ	M	
		N	
		x_0	

   (b) Use the change of variable:

$$y = \frac{x}{N}, \quad s = kt.$$

   Derive the new equation (including the initial condition) in the new variables y and s.

3. Consider a population model:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{N}\right) - \frac{\beta x^2}{m + x}, \tag{2}$$

where $k, N, \beta, m > 0$ are parameters. Use the change of variable:

$$y = \frac{x}{m}, \; s = \beta t.$$

Derive the equation in the new variables.

Convexity of the solutions to population models

A function $f(x)$ is *concave up* if $f''(x) \geq 0$, and $f(x)$ is *concave down* if $f''(x) \leq 0$. For the Malthus model

$$\frac{dP}{dt} = kP \tag{3}$$

the solution $P(t) = P(0)e^{kt}$ is concave up for all $t \in (-\infty, \infty)$. For a solution $P(t)$ of

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right), \tag{4}$$

the logistic equation, by taking derivative, we obtain

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt}\left(1 - \frac{2P}{N}\right).$$

Thus dP/dt achieves the maximum value at $P = N/2$, and $P = N/2$ is also a inflection point where P'' changes sign. When $0 < P < N/2$, the function $P(t)$ is concave up and when $N/2 < P < N$, $P(t)$ is concave down. Hence the convexities of the solutions of malthus model and logistic model are same when $0 < P < N/2$, and for that time period, the growth rate of the population is increasing, and the population grows exponentially.

Next we consider the population model with Allee effect:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right), \tag{5}$$

where $M < 0 < -M < N$. We assume that N is much larger than $|M|$, since an environment can often support a large group (say, 10,000) of an animal species, and only when the species has a very small group, (say, less than 10), it has difficulty to breed. By taking the derivative, we obtain

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt}\left(1 - \frac{2P}{N} - \frac{2P}{M} + \frac{3P^2}{MN}\right).$$

The inflection points of $P(t)$ satisfy

$$3P^2 - (2M + 2N)P + MN = 0.$$

Thus

$$P_1 = \frac{M + N + \sqrt{M^2 - MN + N^2}}{3}, \;\; P_2 = \frac{M + N - \sqrt{M^2 - MN + N^2}}{3}$$

are the roots of the equation. Since M is negative, P_2 is negative, and P_1 is positive. When N is much larger than $|M|$, we can ignore M is the formula of P_1, so

$$P_1 \approx \frac{N + \sqrt{N^2}}{3} = \frac{2N}{3}.$$

Therefore in this case When $0 < P < P_1$, the function $P(t)$ is concave up and when $P_1 < P < N$, $P(t)$ is concave down. Since P_1 is close to $2N/3$, then the convexity of this solution is different from the one of logistic equation.