Math 410(510) Notes (4)
Qualitative behavior of Linear systems
a complete list

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A. Two different positive eigenvalues

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x+2y \\ x+3y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 4$

Eigenvectors: $V_1 = (2, -1)$, $V_2 = (1, 1)$

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

1. $(0,0)$ is the only equilibrium point, and any non-zero solution satisfies $\lim_{t\to\infty} \mathbf{Y}(t) = \infty$ and $\lim_{t\to-\infty} \mathbf{Y}(t) = (0,0)$.

2. There are two linear independent straight line solutions:

$$\mathbf{Y_1} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t, \quad \mathbf{Y_2} = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

which are on the lines $y = -0.5x$ and $y = x$ respectively.

3. The non-straight-line solution satisfies
(i) when $t \to -\infty$, the solution is tangent to the straight line solution on $y = -0.5x$,
(ii) when $t \to \infty$, the solution is almost parallel to the straight line solution on $y = x$.

5. This type of equilibrium point is a (2-dimensional) **source**.

B. Two different negative eigenvalues

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -2x-2y \\ -x-3y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -4$

Eigenvectors: $V_1 = (2, -1)$, $V_2 = (1, 1)$

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$

1. $(0,0)$ is the only equilibrium point, and any non-zero solution satisfies $\lim_{t\to-\infty} \mathbf{Y}(t) = \infty$ and $\lim_{t\to\infty} \mathbf{Y}(t) = (0,0)$.

2. There are two linear independent straight line solutions:

$$\mathbf{Y_1} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t}, \quad \mathbf{Y_2} = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$$

which are on the lines $y = -0.5x$ and $y = x$ respectively.

3. The non-straight-line solution satisfies
(i) when $t \to \infty$, the solution is tangent to the straight line solution on $y = -0.5x$,
(ii) when $t \to -\infty$, the solution is almost parallel to the straight line solution on $y = x$.

5. This type of equilibrium point is a (2-dimensional) **sink**.

C. One negative eigenvalue and one positive eigenvalue

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x + 3y \\ x - y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = -2$

Eigenvectors: $V_1 = (3, 1)$, $V_2 = (1, -1)$

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$

1. $(0, 0)$ is the only equilibrium point.

2. There are two linear independent straight line solutions:

$$\mathbf{Y_1} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t}, \quad \mathbf{Y_2} = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

which are on the lines $y = -0.5x$ and $y = x$ respectively.

3. $\mathbf{Y_1}$ is the only solution which satisfies $\lim_{t \to -\infty} \mathbf{Y}(t) = \infty$ and $\lim_{t \to \infty} \mathbf{Y}(t) = (0, 0)$. (stable solution)

4. $\mathbf{Y_2}$ is the only solution which satisfies $\lim_{t \to \infty} \mathbf{Y}(t) = \infty$ and $\lim_{t \to -\infty} \mathbf{Y}(t) = (0, 0)$. (unstable solution)

5. The non-straight-line solution satisfies
(i) $\lim_{t \to \pm\infty} \mathbf{Y}(t) = \infty$
(ii) when $t \to \infty$, the solution tends to the unstable solution,
(ii) when $t \to -\infty$, the solution tends to the stable solution.

6. This type of equilibrium point is a (2-dimensional) **saddle**.
(there is no saddle in 1-dimension)

D. complex eigenvalues $a \pm bi$, $a < 0$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -2x - 3y \\ 3x - 2y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$

Eigenvectors: $V_1 = (i, 1)$, $V_2 = (-i, 1)$

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} e^{-2t}$

1. $(0, 0)$ is the only equilibrium points, and any non-zero solution satisfies $\lim_{t \to -\infty} \mathbf{Y}(t) = \infty$ and $\lim_{t \to \infty} \mathbf{Y}(t) = (0, 0)$.

2. There is no straight line solutions.

3. Any non-zero solution spiral toward the origin, around the origin infinitely many times.

4. The solution curves $(t, x(t))$, $(t, y(t))$ are decaying periodic functions.

5. This type of equilibrium point is a (2-dimensional) **spiral sink**.

6. Orientation: clockwise or counterclockwise?

Qualitative behavior of solutions:

E. complex eigenvalues $a \pm bi$, $a > 0$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ -3x + 2y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$

Eigenvectors: $V_1 = (-i, 1)$, $V_2 = (i, 1)$

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} e^{2t}$

1. $(0, 0)$ is the only equilibrium points, and any non-zero solution satisfies $\lim_{t\to\infty} \mathbf{Y}(t) = \infty$ and $\lim_{t\to-\infty} \mathbf{Y}(t) = (0, 0)$.

2. There is no straight line solutions.

3. Any non-zero solution spiral away from the origin, around the origin infinitely many times.

4. The solution curves $(t, x(t))$, $(t, y(t))$ are exponential growing periodic functions.

5. This type of equilibrium point is a (2-dimensional) **spiral source**.

6. Orientation: clockwise or counterclockwise?

Qualitative behavior of solutions:

F. complex eigenvalues $a \pm bi$, $a = 0$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} y \\ -2x \end{pmatrix}$$

Eigenvalues: $\lambda_1 = \sqrt{2}i$ and $\lambda_2 = -\sqrt{2}i$

Eigenvectors: $V_1 = (1, \sqrt{2}i)$, $V_2 = (1, -\sqrt{2}i)$

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(\sqrt{2}t) \\ -\sin(\sqrt{2}t) \end{pmatrix}$

1. $(0, 0)$ is the only equilibrium points, and any non-zero solution is a periodic solution.

2. There is no straight line solutions.

3. Any non-zero solution spiral around the origin infinitely many times, but stay on a periodic orbit.

4. The solution curves $(t, x(t))$, $(t, y(t))$ are periodic functions.

5. This type of equilibrium point is a (2-dimensional) **center**.

6. Orientation: clockwise or counterclockwise?

G. Two real eigenvalues: $\lambda_1 > \lambda_2 = 0$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 4x + 6y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = 8$

Eigenvectors: $V_1 = (3, -2)$, $V_2 = (1, 2)$

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{8t}$

1. Any point on the line $2x+3y = 0$ is an equilibrium point, and any non-equilibrium solution satisfies $\lim_{t\to\infty} \mathbf{Y}(t) = \infty$ and $\lim_{t\to-\infty} \mathbf{Y}(t)$ is an equilibrium.

2. Any non-equilibrium solution is a straight line solution.

3. This type of equilibrium point is a (2-dimensional) **degenerate source**.

H. Two real eigenvalues: $0 = \lambda_1 > \lambda_2$

degenerate sink

I. Repeated eigenvalues: $\lambda_1 = \lambda_2 > 0$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = \lambda = 2$

Eigenvectors: $V_1 = (1, 0)$, $V_2 = (0, 1)$ or any vector

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$

1. $(0, 0)$ is the only equilibrium point, and any non-zero solution satisfies $\lim_{t\to\infty} \mathbf{Y}(t) = \infty$ and $\lim_{t\to-\infty} \mathbf{Y}(t) = (0, 0)$.

2. Each solution is a straight line solution, and any ray from $(0, 0)$ is a solution orbit.

3. This type of equilibrium point is a (2-dimensional) **star source**.

J. Repeated eigenvalues: $\lambda_1 = \lambda_2 < 0$

star sink

K. Repeated eigenvalues: $\lambda_1 = \lambda_2 > 0$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x + y \\ 2y \end{pmatrix}$$

Eigenvalues: $\lambda_1 = \lambda = 2$

Eigenvectors: $V_1 = (1, 0)$,

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \left(t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) e^{2t}$

1. $(0, 0)$ is the only equilibrium point, and any non-zero solution satisfies $\lim_{t\to\infty} \mathbf{Y}(t) = \infty$ and $\lim_{t\to-\infty} \mathbf{Y}(t) = (0, 0)$. There is one straight line solution $\mathbf{Y_1} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$, which is on the line $y = 0$.

2. Any non-straight-line solution satisfies (i) $t \to -\infty$, the solution is tangent to the straight line solution, (ii) $t \to \infty$, the solution is almost parallel to the straight line solution (but in the opposite direction).

3. This type of equilibrium point is a (2-dimensional)
trying to spiral source.

L. Repeated eigenvalues: $\lambda_1 = \lambda_2 < 0$

trying to spiral sink

M. Repeated eigenvalues: $\lambda_1 = \lambda_2 = 0$

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = \lambda = 0$

Eigenvectors: $V_1 = (1, 0)$,

General solution: $\mathbf{Y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$

1. Any point on $y = 0$ is an equilibrium point, any non-equilibrium solution is a straight line solution which is parallel to the line of equilibrium points.

2. This type of equilibrium point is a (2-dimensional)
parallel lines.

N. Repeated eigenvalues: $\lambda_1 = \lambda_2 < 0$: $a = b = c = d = 0$

dumb system

<u>Classification:</u>

Two real eigenvalues:

1. $\lambda_1 > \lambda_2 > 0$: source
2. $\lambda_1 > \lambda_2 = 0$: degenerate source
3. $\lambda_1 > 0 > \lambda_2$: saddle
4. $\lambda_1 = 0 > \lambda_2$: degenerate sink
5. $0 > \lambda_1 > \lambda_2$: sink

Two complex eigenvalues: $\lambda_\pm = a \pm bi$

1. $a > 0$: spiral source
2. $a = 0$: center
3. $a < 0$: spiral sink

One real eigenvalue: $\lambda_1 = \lambda_2 = \lambda$

1. $\lambda > 0$: star source or trying to spiral source
2. $\lambda = 0$: parallel lines or dumb system
3. $\lambda < 0$: star sink or trying to spiral sink

Stable equilibrium: sink, spiral sink, star sink or trying to spiral sink (negative eigenvalues, or complex eigenvalues with negative real part)

Others: unstable

(neutrally stable: degenerate sink, degenerate source, center)