Problem Set 3
Recurrent Sequences (and some Pigeonhole principle)
Discussion: Oct. 5th, 2004

A recurrent sequence \( \{x_n\} \) is defined as
\[
x_{n+k} = f(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}), \quad x_0 = a_0, \ x_1 = a_1, \ldots, \ x_{k-1} = a_{k-1}.
\]
(1) is also called a difference equation. The number \( k \) is the order of the equation (or relation). A \( k \)-th order linear recurrent sequence is generated by a linear equation:
\[
x_{n+k} = b_n x_n + b_{n+1} x_{n+1} + b_{n+2} x_{n+2} + \cdots + b_{n+k-1} x_{n+k-1},
\]
\[
x_0 = a_0, \ x_1 = a_1, \ldots, \ x_{k-1} = a_{k-1}.
\]
(2)

The most common ones are first order recurrent sequence:
\[
x_{n+1} = f(n, x_n), \ x_0 = a_0,
\]
(3) or second order recurrent sequence:
\[
x_{n+2} = f(n, x_n, x_{n+1}), \ x_0 = a_0, \ x_1 = a_1.
\]
(4)

An autonomous first order recurrent sequence:
\[
x_{n+1} = f(x_n), \ x_0 = a_0,
\]
(5) is also often called a map. The theory of linear recurrent sequence is very similar to that of linear ordinary differential equation. For example, a second order linear recurrent sequence:
\[
x_{n+2} = Ax_n + Bx_{n+1}, \ x_0 = a_0, \ x_1 = a_1.
\]
(6)

The solution is given by
\[
x_n = c_1 \lambda_1^n + c_2 \lambda_2^n,
\]
(7) where \( \lambda_1 \) and \( \lambda_2 \) are the roots of quadratic characteristic equation \( \lambda^2 = A\lambda + B \), and \( c_1 \), \( c_2 \) are to be determined by the initial conditions. This is similar to the solution of second order differential equation:
\[
y'' = Ay' + By, \ y(0) = a_0, \ y'(0) = a_1,
\]
(8) for which the solution is
\[
y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},
\]
(9) where \( \lambda_1 \) and \( \lambda_2 \) are the roots of quadratic characteristic equation \( \lambda^2 = A\lambda + B \).
Exercises:
1. Find general solution of 1st order equation: \( x_{n+1} = Ax_n \), \( x_0 = a_0 \);
2. Find general solution of 1st order equation: \( x_{n+1} = Ax_n + C \), \( x_0 = a_0 \);
3. Find general solution of 2nd order equation: \( x_{n+2} = Ax_{n+1} + Bx_n + C \), \( x_0 = a_0 \), \( x_1 = a_1 \);
4. Find the solution of Fibonacci sequence: \( x_{n+2} = x_{n+1} + x_n \), \( x_0 = x_1 = 1 \).

Since linear autonomous recurrent equations always have solution formulas, most problems in mathematics competitions are either non-autonomous or nonlinear. However the methods for linear equation are still very useful, and sometimes non-autonomous or nonlinear maybe reduced to linear autonomous equation via certain smart change of variables. In general there is no explicit solution formula for non-autonomous or nonlinear equation, even for a simple equation like \( x_{n+1} = Ax_n(1-x_n) \) (Logistic equation). Indeed the solutions of logistic equation are chaotic when the parameter \( A \) is large. (If you have the textbook of Math 302 (Blanchard-Devaney-Hall: Differential Equations), Chapter 8 of that book has a good introduction for logistic equation.)

For the 1st order nonlinear autonomous recurrent equation \( x_{n+1} = f(x_n) \), a fixed point \( x \) is the one satisfying \( x = f(x) \). Note that for such equation, \( x_{n+1} = f^n(x_0) \), where \( f^n(y) = f(f^{n-1}(y)) \). So it is also often called iterated sequence. A fixed point \( x \) is attracting if \( f^n(y) \to x \) for all \( y \) near \( x \), and it is repelling if \( f^n(y) \) goes away from \( x \) for all \( y \) near \( x \). (There are also fixed point neither attracting nor repelling.) A fixed point \( x \) is attracting if \( |f'(x)| < 1 \), and it is repelling if \( |f'(x)| > 1 \). The iterated sequence ca be drawn in \( x-y \) coordinate system with so-called web diagram.

Finally, one can have a system of difference equation:

\[
\begin{align*}
x_{n+1} &= Ax_n + By_n, \quad y_{n+1} = Cx_n + Dy_n,
\end{align*}
\]

and the solution is given in a form \( (x_n, y_n) = (c_1, c_2)\lambda_1^n + (c_3, c_4)\lambda_2^n \), and \( \lambda_1, \lambda_2 \) are the eigenvalues of the matrix

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

Some web-links on difference equations:

http://hypatia.math.uri.edu/~kulem/diffeqaturi/dehomepage.html
http://www.math.duke.edu/education/ccp/materials/linalg/diffeqs/contents.html

Problems on recurrent relations:
(and the last four are about pigeonhole principle)

1. (UIUC 2004) Define a sequence \( \{a_n\} \) by \( a_0 = 0 \), \( a_1 = 1 \), \( a_2 = 2 \), and \( a_n = a_{n-1} + a_{n-2} - a_{n-3} + 1 \) for \( n \geq 3 \). Find, with proof, \( a_{2004} \).

2. (UIUC 1999) Define a sequence \( \{x_n\} \) by \( x_1 = \sqrt{2} \), and \( x_{n+1} = \sqrt{2}x_n \) for \( n \geq 1 \). Prove the sequence \( \{x_n\} \) converges and find its limit.
3. (UIUC 1998) A sequence $a_0, a_1, a_2, \ldots$ of real numbers is defined recursively by $a_0 = 1$, $a_{n+1} = \frac{a_n}{1+na_n}$, $n = 0, 1, 2, \ldots$. Find a general formula of $a_n$.

4. (UIUC 1997) Let $x_1 = x_2 = 1$, and $x_{n+1} = 1996x_n + 1997x_{n-1}$ for $n \geq 2$. Find (with proof) the remainder of $x_{1997}$ upon division by 3.

5. (UIUC 1997) Let $x_0 = 0$, $x_1 = 1$, and $x_{n+1} = \frac{x_n + nx_{n-1}}{n+1}$ for $n \geq 1$. Show that the sequence $\{x_n\}$ converges and finds its limit.

6. (UIUC 2003 Mock) Let $f(x) = \frac{1}{1-x}$. Let $f_1(x) = f(x)$ and for each $n = 2, 3, \ldots$, let $f_n(x) = f(f_{n-1}(x))$. What is the value of $f_{2003}(2003)$?

7. (UIUC 2003 Mock) Given $x_0 = 0$, define $x_{k+1} = \frac{x_k^2 - 2}{2x_k - 3}$. Determine if the sequence $\{x_n\}$ is convergent and if it is, find its limit.

8. (UIUC 1995) Let $c$ be a positive constant, let $0 < x_1 < x_0 < 1$, and for $n \geq 1$ let $x_{n+1} = cx_nx_{n-1}$. Prove that there exists a positive real number $\alpha$ such that the limit $L = \lim_{n \to \infty} \frac{x_{n+1}}{x_n^\alpha}$ exists and $0 < L < \infty$.

9. (Putnam 1966) Let $0 < x_0 < 1$, and $x_{n+1} = x_n(1-x_n)$ for $n \geq 0$. Prove that the limit $\lim_{n \to \infty} nx_n$ exists and is equal to 1.

10. (Stanford) Prove that there is some integer power of 2 that begins 2002· · ·.

11. (UIUC 1996) Let $a_1 < a_2 < \cdots < a_{43} < a_{44}$ be positive integers not exceeding 125. Prove that among the 43 differences $d_i = a_{i+1} - a_i$ $(i = 1, 2, 3, \cdots, 43)$ some value must occur at least 10 times.

12. (UIUC 2000) Suppose that $a_1, a_2, \ldots, a_n$ are $n$ given integers. Prove that there exist integers $r$ and $s$ with $0 \leq r < s \leq n$ such that $a_{r+1} + a_{r+2} + \cdots + a_s$ is divisible by $n$.

13. (UIUC 2002) Let $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, and for $n \geq 4$ define $a_n$ to be the last digit of the sum of the proceeding three terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) 24846828· · · Determine, with proof, whether or not the string 2002 occurs somewhere in the sequence.